RH, NSE, YME, GUT solutions

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This homepage addresses the three Millennium problems

A. The Riemann Hypothesis (RH)
B. The 3D-Navier-Stokes equations (NSE) navier-stokes-equations.com
C. The Yang-Mills equations (YME) quantum-gravitation.de

building on a commonly applied mathematical framework. The proposed solutions are building on common mathematical solution concepts and tools, e.g. Hilbert scale, Hilbert (resp. Riesz) transform(s), Hilbert-Polya conjecture. The proposed mathematical framework is also suggested to build a unified quantum field and gravity field theory, while applying the proposed mathematical framework to the

D. Loop Quantum Theory (LQT) C. Rovelli (RoC), (RoC1)

The LQT is a modern version of the Wheeler/deWitt theory ((CiI), (WhJ)). The H(-1/2) Hilbert space of this page is proposed as extended "LQT spin network state" Hilbert space, enabling an (**unified** Dirac+Yang-Mills+Higgs+Einstein) **hamiltonian operator** ((RoC) 7.2.3), including the 4th matter **plasma state**. It is known from general relativity and quantum theory that all of them are fakes resp. interim specific mathematical model items.

As the LQT topic has been added only recently to this page, we briefly sketch the central **conceptual differentiators of LQT** to other GUT theory attempts ((RoC) p. 10, p. 14, p. 140) with its relationship to the topic of this page (related conceptual elements of the Wheeler theory are sketched below):

(RoC1): "*the **key differentiator** to **Einstein's field theory** is the **absence of** the familar **"space-time" stage** (**background independence**), which is technically realized by the gauge invariance of the action under (active) diffeomorphisms (or **diffeomorphism invariance**). It is the combination of two properties of the action: its invariance under arbitrary changes of coordinates and the fact that there is **no nondynamical "background" field**. ... The notions of space and time is given up. The **space** continuum "on which" things are located and the **time** "along which" evolution happens are **semiclassical approximation** notions; the LQT makes only use of the general tools of quantum theory: a Hilbert space of states, operators related to the measurements of physical quantities, and transition amplitudes that determine the probability outcome of measurements of these quantities. ... In the macroscopic world, the physical variable t measured by a clock has peculiar properties. The fact, that time is not a special variable at the fundamental level needs to be reconciled, leading to*

the thermal time hypothesis (RoC1): "*In Nature, there is no preferred physical time variable t. There are no equilibrium state "r" preferred a priori. Rather, all variables are equivalent: we can find the system in an arbitrary state "r"; if the system is in a state "r", then a preferred variable is singled out by the state of the system. This variable is what we call time*."

The three conceptual elements of the quantum mechanics (remaining in LQT) are (RoC1): "*(1)* ***granularity*** *(2)* ***indeterminism*** *(3)* ***fluctuation*** */relations of physical variables resp. phenomena. With respect to the physical phenomenon "time" this means that for all physical phenoma there is (1) granularity: a smallest "time" unit, the Planck time (2) indeterminism: quantum super position of time (3) fluctuation (Heisenberg), when trying to determine the position of an electron today and tomorrow*". In quantum mechanics "**time**" and "**energy**" are conjugated variables linked by the concept of "**action**" ((HeW) II, 2c). Therefore, the conceptual elements above find its counterpart with respect to the proposed quantum state Hilbert space H(-1/2) by the facts, that the Hilbert sub-space **L(2)=H(0)** is **compactly embedded into H(-1/2),** where physical quantum mechanics phenomena are "measured" by corresponding **hermitian (projection) operators onto L(2)**. This property is proposed as the mathematical model for quantum mechanics "granularity" states in H(0). At the same point in time this embedded "granular" Hilbert space (with respect to the norm of H(-1/2)) is the standard **L(2)** framework of **probability theory**, statistical analysis and quantum mechanics. An analogue situation is given by the rational numbers (~H(0)) as subset of the real or hyperreal numbers (~H(-1/2)): the rational numbers are embedded into the **ordered field** of **real numbers**, **which is a subset of** the **ordered field** of **hyper-real (ideal) numbers**. The field of hyper-real numbers (or ideal points) contains infinitely great and small numbers. It is constructed abstractly using Zorn's lemma. The key differentiator of both fields is, that the **Archimedian axiom** (which is valid for the real numbers) is no longer valid for the ordered hyper-real numbers.

From a purely mathematical point of view the baseline of all mathematical models are "axioms"; the very first one to be mentioned in the context of the above is the "**well-ordering theorem**" (which is NOT a "theorem" as such). It is equivalent to the "**axiom of choice**" and "**Zorn's lemma**" (which is NOT a "lemma", as such). At the same time all "physical" gravity and quantum theory models are a purely mathematical models building on those kind of axioms. With respect to an appropriate definition of a "mathematical time" beyond the ""physical"/ thermodynamical time" ((RoC) III.9) one could decide for a hyper-real number (which is nothing else than a Leibniz monad), where the corresponding standard part of it (in case the hyper-real number is finite) is the "thermodynamical time" variable. If this option is beeing seen as too sophisticated, please note that already each **irrational number** is its own mystery/universe, as EACH irrational number is only "existing" (i.e. purely mathematically defined) as the limit of a sequence of INFINITE rational numbers.

The last section of (RoC1) is related to philosophical aspects (including the words of Anaximander, (HeM) "Der Spruch des Anaximander"). From (HeM), Die Zeit des Weltbildes, 72) we recall the following: "*Die neuzeitliche Physik heisst mathematische, weil sie in einem vorzüglichen Sinne eine ganz bestimmte Mathematik anwendet. Allein, sie kann in solcher Weise nur mathematisch verfahren, weil sie in einem tieferen Sinne bereits mathematisch ist. .... Keineswegs wird aber das Wesen des Mathematischen durch das Zahlenhafte bestimmt. ... Wenn nun die Physik sich ausdrücklich zu einer mathematischen gestaltet, dann heisst das: Durch sie und für sie wird in einer betonten Weise etwas als das Schon-Bekannte im vorhinein ausgemacht. Dieses Ausmachen betrifft nichts Geringeres als den Entwurf dessen, was für das gesuchte Erkennen der Natur künftig Natur sein soll: der in sich geschlossene* ***Bewegungszusammenhang raum-zeitlicher Massenpunkte***."

Solution(s) concept(s) walkthrough

In order to prove the **Riemann Hypothesis** (RH) the Polya criterion can not be applied in combination with the Müntz formula ((TiE) 2.11). This is due to the divergence of the Müntz formula in the critical stripe due to the asymptotics behavior of the baseline function, which is the **Gaussian function**. The conceptual challenge (not only in this specific case) is about the not vanishing constant Fourier term of the Gaussian function and its related impact with respect to the Poisson summation formula. The latter formula applied to the Gaussian function leads to the Riemann duality equation ((EdH) 1.7). A proposed alternative "baseline" function than the Gaussian function, which is its related Hilbert transform, the **Dawson function**, addresses this issue in an alternative way as Riemann did. As the Hilbert transform is a convolution integral in a correspondingly defined distributional Hilbert space frame it enables the **Hilbert-Polya conjecture** (e.g. (CaD))**.** The corresponding distributional ("periodical") Hilbert space framework, where the Gaussian / Dawson functions are replaced by the fractional part / log(2sin)-functions enables the **Bagchi** reformulation of the **Nyman-Beurling RH criterion**.

The corresponding formulas, when replacing the Gaussian function by its Hilbert transform, are well known: the **Hilbert transform** of the **Gaussian** is given by the **Dawson integral** (GaW). Its properties are e.g. provided in ((AbM) chapter 7, (BrK4) lemma D1). The Dawson function is related to a special **Kummer function** in a similar form than the (error function) erf(x)-function resp. the li(x)-function ((AbM) (9.13.1), (9.13.3), (9.13.7), (LeN) 9.8, 9.13). A characterization of the Dawson function as an sin-integral (over the positive x-axis) of the Gaussian function is given in ((GrI) 3.896 3.). Its Mellin transform is provided in ((GrI) 7.612, (BrK4) lemma S2). The **asymptotics of** the **zeros** of those degenerated hypergeometric functions are given in (SeA) resp. ((BrK4) lemma A4). The **fractional part function** related Zeta function theory is provided in ((TiE) II).

With respect to the considered distributional Hilbert spaces H(-1/2) and H(-1) we note that the Zeta function is an integral function of order 1 and an element of the distributional Hilbert space H(-1). This property is an outcome of the relationship between the Hilbert spaces above, the **Dirichlet series** theory (HaG) and the Hardy space isometry as provided in e.g. ((LaE), §227, Satz 40). With respect to the physical aspects below we refer to (NaS), where the H(1/2) dual space of H(-1/2) on the circle (with its inner product defined by a **Stieltjes integral**) is considered in the context of **Teichmüller theory** and the universal period mapping via **quantum calculus**. For the corresponding Fourier series analysis we refer to ((ZyA) XIII, 11). The approximation by polynomials in a complex domain leads to several notions and theorems of convergence related to **Newton-Gaussian** and **cardinal series**. The latter one are closely connected with certain aspects of the theory of Fourier series and integrals. Under sufficiently strong conditions the cardinal function can be resolved by Fourier's integral. Those conditions can be considerably relaxed by introducing Stieltjes integrals resulting in **(C,1) summable series** ((WhJ1) theorems 16 & 17, (BrK4) remarks 3.6/3.7).

The **RH** is connected to the **quantum theory** via the Hilbert-Polya conjecture resp. the **Berry-Keating conjecture**. It is about the hypothesis, that the imaginary parts t of the zeros 1/2+it of the **Zeta function** Z(t) corresponds to eigenvalues of an **unbounded self-adjoint operator**, which is an appropriate Hermitian operator basically defined by QP+PQ, whereby Q denotes the location, and P denotes the (Schrödinger) momentum operator. In (BrK3) the corresponding model (convolution integral) operator S(1) (of order 1 with "density" d(cotx)) for the one-dimensional harmonic quantum oscillator model is provided. The theory of spectral expansions of non-bounded self-adjoint operator is connected with the notions "Lebesgue-**Stieltjes integral**" and "**functional Hilbert equation for resolvents** ((LuL) (7.8).The corresponding Hilbert scale framework plays also a key role on the inverse problem for the double layer potential.

The corresponding model problem (w/o any compact disturbance operator) with the Newton kernel enjoys a **double layer potential integral operator** with the eigenvalue 1/2 (EbP).

The Riemann entire Zeta function Z(s) enjoys the functional equation in the form Z(s)=Z(1-s). The alternatively proposed Dawson (baseline) function leads to an alternative entire Zeta function definition **Z(*;s)** with a corresponding **functional equation** in the form **Z(*,1-s) = Q(s) * Z(*,s)**, with **Q(s):=P(s)/P(1-s)**, whereby **P(x):=cx*cot(cx)** and the constant c denotes the number "pi"/2. Therefore, the alternative entire Zeta function definition **Z(*;s)** have same nontrivial zeros as Riemann's entire Riemann Zeta function Z(s).

The RH is equivalent to the **Li criterion** governing a sequence of real constants, that are certain logarithmic derivatives of Z(s) evaluated at unity (LiX). This equivalence results from a necessary and sufficient condition that the logarithmic of the function Z(1/(1-z)) be analytic in the unit disk. The proof of the Li criterion is built on the two representations of the Zeta function, its (product) representation over all its nontrivial zeros ((HdE) 1.10) and Riemann's integral representation derived from the Riemann duality equation, based on the **Jacobi theta function** ((EdH) 1.8). Based on Riemann's integral representation involving Jacobi's theta function and its derivatives in (BiP) some particular probability laws governing sums of independent exponential variables are considered. In (KeJ) corresponding Li/Keiper constants are considered. The proposed **alternative entire Zeta function Z(*,s)** is suggested to derive an analogue Li criteron.

One proof of the Riemann functional equation is based on the **fractional part function** r(x), whereby the zeta function *zeta(s)* in the critical stripe is given by the Mellin transform *zeta*(1-s) = M(-x*d/dx(r(x))(s-1) ((TiE) (2.1.5). The functional equation is given by ***zeta*(s) = *chi*(s)*zeta(1-s)**, whereby *chi*(s) is defined according to ((TiE) (2.1.12). The Hilbert transform of the fractional part function is given by the log(sin(x))-function. After some calculations (see also (BrK4) lemma 1.4, lemma 3.1 (GrI) 1.441, 3.761 4./9., 8.334, 8.335) the corresponding **alternative zeta(*,s) function** is given by ***zeta*(*,1-s) * s = *zeta*(1-s) * tan(c*s)**.

The **density function J(x)** of the **log(*zeta*(s))** Fourier inverse integral representation can be reformulated into a representation of the function pi(x) (that is, for the "number of primes counting function" less than any given magnitude x ((EdH) 1.17)). Riemann's proof of the formula for J(x) results into the famous Riemann approximation error function ((HdE) 1.17 (3)) based on the product formula representation of the Gamma function G(1+s/2) ((HdE) 1.3 (4), (GrI) 8.322). The challenge to prove the corresponding **li(x) function approximation criterion** (i.e. li(x) - pi(x)=O(log(x)*squar(x)) = O(x*exp(1/2+e)), e>0, (BrK4) p.10) is about the (exponential) asymptotics of the Gaussian function ((EdH) 1.16, (BrK4) note S25). In this context we note that the **Dawson** function enjoys an only **polynomial asymptotics** in the form O(x*exp(-1)).

In summary, the alternatively proposed Gamma G(*,s/2) function leads to an **alternative Riemann approximation error function** with improved convergence behavior (at least with respect to the proposed Hilbert space norms). The appreciated asymptotics of the Dawson function suggested an **alternative li(*,x) function** definition, whereby, of course, the result of Chebyshev about the proven relative error in the approximation of pi(x) by Gauss' li(x) function needs to be taken into account ((EdH) 1.1 (3)). Alternatively to the **Gaussian density** dg=log(1/t)dt the above indicates to consider the **Clausen density** dw, where w(t) denotes the periodical continuation of the Clausen integral ((AbM) 27.8). Obviously the Clausen integral is related to the Hilbert transform of the fractional part function.

The **Dawson function F(x)** (i.e. the Hilbert transform of the Gaussian function f(x):=exp(-(x*x))) is related to the two special **Kummer functions K(1;z):=K(1,3/2;z)** and **K(2;z):=K(1/2,3/2,z)** by **F(x) = x*K(1;-x*x)** ((LeN) (9.13.3)) resp. **F(x) = x * f(x) * K(2,x*x)** ((GrI), 9.212). It provides an option to replace the auxiliary functions G(b) resp. E(b) in (EdH) 1.14, 1.16, to derive the formula for the Riemann density function J(x) ((EdH) 1.12 (2)). Both special Kummer functions enjoy appreciated **non-asymptotics** of its **zeros** (SeA): let c="pi" denote the unit circle constant, then the imaginary part of the zeros of both functions fulfill the inequality **(2n-1)*c<abs(Im(z(n)))<2n*c**, while the real parts fulfill Re(z)<-1/2 resp. Re(z)>1/2 for K(1;z) resp. K(2;z). In other words, there are no zeros of K(2;z) on the critical line s=1/2*it (t ex R), resp. there are no zeros of K(1;z) on the "dual" line (1-s).

The density of prime numbers appears to be the **Gaussian density** dg=log(1/t)dt defining the corresponding prime number counting integral function ((EdH) 1.1 (3)). We mention the Kummer function based representation of the li-function in the form li(x)=-x*K(1,1;-logx) ((LeN) (9.13.7)). Let G(x) denote the first derivative of K(2;z), i.e. dK(2;x)/dx, then the series representation of the exponential function exp(x) is asymptotically identical to x*G(x) ((BrK4), lemma K2). By substitution of the integration variable by t --> exp(y) this results into an alternative prime number approximation function in the form **2*K(2,logx)** =2*(x - logx * G(logx)) = 2*(x - logx * (1/3) * K(3/2,5/2;logx)).

The relationship of the considered Kummer functions to the incomplete Gamma function is provided in (AbM) 6.5.12. For the asymptotics of the Kummer functions we refer to (OlF), 7 §10.1, (AbM) 13.5.1. We further note, that the **generalized asymptotic (Poincaré) expansion** admits expansions that have no conceivable value, in an analytical or numerical sense, concerning the functions they represent. In (OlF) §10, the expansion of sin(x)/x is provided with first summand term exp(-x)/logx.

Additionally, the above alternative Z(s) resp. *zeta*(s) function representations indicate an alternative Gamma (auxiliary) function definition in the form G(*,s/2):=G(s/2)*tan(cs)/s with identical asymptotics for x --> 0. Corresponding formulas for the tan(x)- resp. the **log(tan)-function** are provided in ((GrI) 1.421,1.518). In (ElL) the **Fourier expansion** of the log(tan) function is provided, giving a note to its related Hilbert space H(a) regularity. Its graph looks like a beautiful white noise diagram. In (EsO), formulas (6.3), (6.4), the Fourier expansion of log(Gamma(x)) function is provided with coefficients a(n)=1/(2n), b(n)=(A+logn)/(2cn) and a(0)=log(root(4c)).

For other related application areas of G(*,s/2) we refer to Ramanujan's chapter "Analogues of the Gamma Function" ((BeB chapter 8).

In (TiE) theorem 4.11, an approximation to the *zeta* function series in the critical stripe by a partial sum of its **Dirichlet series** is given ((BrK4) remark 3.8). One proof of this theorem is built on a simple application of the theorem of residues, where the *zeta* series is expressed as a (Mellin transform type) **contour integral** of the **cot(cz)-function** ((TiE) 4.14). As the cot and the *zeta* function are both elements of the distributional Hilbert space H(-1) the contour integral above with a properly chosen contour provides a contour integral representation for the *zeta* in a weak H(-1) sense. In (ChK) VI, §2, two expansions of cot(z) are compared to prove that all coefficients of one of this expansion (zeta(2n)/pi(exp(2n))) are rational. Corresponding formulas for odd inters are unknown. In (EsR), example 78, a **"finite part"-"principle value" integral representation** of the **c*cot(cx)** is given (which is zero also for positive or negative integers). It is used as enabler to obtain the asymptotic expansion of the p.v. integral, defined by the "restricted" Hilbert transform integral of a function u(x) over the positive x-axis, only. In case u(x) has a structure u(x)=v(x)*squar(x) the representation enjoys a remarkable form, where the numbers n+1/2 play a key role.

In ((BrK4) lemma 3.4, lemma A12/19) the function P(x) is considered in the context of (appreciated) quasi-asymptotics of (corresponding) distributions ((ViV) p. 56/57) and the Riemann mapping theorem resp. the Schwarz lemma. The considered "function" g(x):=-d/dx(cot(x)) (whereby the cot-"function" is an element of H(-1)) is auto-model (or regular varying) of order -1. This condition and its corresponding asymptotics property ((BrK) lemma 3.4) provide the prereqisitions of the **RH Polya criterion** ((PoG), (BrK5) theorem 6). The above quasi-asymptotics indicates a replacement of the differential d(logx)by d(log(sinx)). The cot(z) function expansions (ChK) VI, §2) in combination with **Ramanujan's formula** ((EdH) 10.10) resp. its generalization theorem ((EdH) p.220) is proposed to be applied to define an appropriate analytical (Mellin transform) function in the stripe 1/2<Re(s)<1.

In (GrI) 8.334, the relationship between the the cot- and the Gamma function is provided. From (BeB) 8. Entry17(iii)) we quote: "*the* **indefinite *Fourier series of the cot(cx)-function*** *may be formally established by differentiating the corresponding Fourier series equation for* (the L(2)=H(0)-function) ***-log(2sin(cx))***" ((BrK4) remark 3.8). The proposed distributional Hilbert scales provide the proper framework to justify Ramanujan's related parenthetical remark "*for the same limit*" (in a H(-1)-sense).

With respect to the **NSE** and the **YME** the proposed mathematical concepts and tools are especially correlated to the names of **Plemelj**, **Stieltjes** and **Calderón**. The essential estimate for the critical non-linear term of the non-linear, non-stationary 3-D NSE has been provided by **Sobolevskii**. With respect to the YME the proposed mathematical concepts and tools are especially correlated to the names of **Schrödinger** and **Weyl** (e.g. in the context of "half-odd integers quantum numbers for the Bose statistics" and resp. Weyl's contributions on the concepts of matter, the structure of the world and the principle of action (WeH), (WeH1), (WeH2)). It enables an **alternative** (quantum) **ground state energy model** embedded in the proposed distributional Hilbert scale frame of this homepage covering all variational physical-mathematical PDE and Pseudo Differential Operator (PDO) equations (e.g. also the Maxwell equations).

The **Dirac theory** with its underlying concept of a "Dirac function" (where the regularity of the Dirac distribution "function" depends from the space dimension) is **omitted** and replaced by a distributional Hilbert space (domain) concept. This alternative concept avoids space dimension depending regularity assumptions for (quantum) physical variational model (wave package) states and solutions (defined e.g. by energy or operator norm minimization problems) and physical problem specific "force" types.

The until today not successfull attempts to define a **quantum gravity model** is about dynamics models **coupling gravity + matter system**, simply defined by adding the terms defining the matter dynamics to the corresponding field related (i.e. Dirac+Yang-Mills+Higgs+Einstein) hamiltonians ((RoC) 7.3)). The best case result (which is unlikely to be achieved anyway) is then about a "four different forces" model (not only four different force type phenomena) governed by the same (transcendental) "energy" based on corresponding energy least action principles (whereby *only "the least action principle in his most modern general public is a* ***maxime of Kant's reflective judgment***"). The ***loop quantum theory*** *(LQT)* (**C. Rovelli**) is the choice of a different algebra of basic field functions: a noncanonical algebra based on the holonomics of the gravitational connections ((RoC) 1.2.1). The holonomy (or the "Wilson loop") is the matrix of the parallel transport along a closed curve and **spacetime** itself is **formed** by **loop-like states**. Therefore the position of a loop state is relevant only with respect to other loops, and not with respect to the background. The **state space** of the theory is a **separable Hilbert space** spanned by loop states, admitting an orthogonal basis of spin network states, which are formed by **finite linear combinations of loop states**, and are defined precisely as the spin network states of a lattice **Yang-Mills theory**."

The proposed distributional quantum state H(-1/2) with corresponding inner product admits and requires *infinite* linear combinations of LQT "loop states" (which we "promoted" becoming "quantum fluid/quantum element/"truly orthogonal fermion & boson/rotating differential/ideal point/monad" states), i.e. it overcomes the **current challenge of LQT** defining the scalar product of the spin network state Hilbert space ((RoC) 7.2.3). The physical LQT (kinematical) space (which is a quantum superposition of the QLT "**spin networks**") corresponds to an orthogonal projection of H(-1/2) onto H(0). This othogonal projection can be interpreted as a general model for a "**spontaneous symmetry break down**".

In the following we briefly sketch the conceptual common solution elements motivating our terminology of a "common Hilbert space framework" to solve the three considered Millenium problems (and a few other related and considered ones).

1. The common Hilbert scale frame & its corresponding common solution idea

The common Hilbert scale is about the Hilbert spaces H(a) with a=1,1/2,0,-1/2,-1 with its corresponding inner products **((**u,v**))**, **(**u,v**)**, (u,v), ((u,v)), (((u,v))).

The **RH** is connected to the **quantum theory** via the Hilbert-Polya conjecture resp. the **Berry-Keating conjecture**. The latter one is about a physical reason, why the RH should be true. This would be the case if the imaginary parts t of the zeros 1/2+it of the **Zeta function** Z(t) corresponds to eigenvalues of an unbounded self-adjopint operator, which is an appropriate **Hermitian operator** basically defined by QP+PQ, whereby Q denotes the location, and P denotes the (Schrödinger) momentum operator. The notion "***unbounded***" is not well defined, as an operator is only well-defined by describing the operator "mapping" in combination with its defined domain. The Zeta function is an **element of H(-1)**, but not an element of H(-1/2).Therefore, there is a characterization of the Zeta function on the critcal line in the form ((Z,v)) for all v ex H(0). As the "test space" H(0) is compactly embedded into H(-1/2) this shows that there is an extended Zeta function Z(*)=Z+Z(#) with the characterization ((Z(*),v)) for all v ex H(-1/2), where Z can be interpreted as orthogonal approximation of Z(*) with discrete spectrum.

The **Gaussian function** f(x) plays a key role in the Zeta function theory, as well as in the quantum theory. Its Mellin transform defines the factor function between the Zeta function and its corresponding entire Zeta function, which builds the **Riemann duality equation**. The Riemann duality equation involves an inner product which is natural with respect to the additive structure of R(+), namely d(log(x))=dx/x, rather than the multiplicative structure, namely the L(2)=H(0) inner product. This structure jeopardizes all attempts so far to represent the entire Zeta function as convolution integral, which would prove the RH. There is a (formally only!!) **self-adjoint operator representation** with transform being the entire Zeta function, but in fact this operator has no transform at all, as the corresponding integral representation does not converge for any complex s ((EdH), 10.3). The root cause is related to the Poisson summation formula in combination with the fact that the constant Fourier term of the Gaussian function is not vanishing. Riemann overcame this challenge by replacing the Gaussian function f(x) by the product of the variable "x" and its first derivative, i.e. x --> h(x):=x*d/dx(f(x)). The corresponding Mellin transform "effect" is about a multiplication with the factor -s, i.e it does not effect the factor (s-1), which is the counterpart of the Li-function.

The **central idea** of our alternative approach is "just" to alternatively **replace** the **Gaussian function by its Hilbert transform** (which is the **Dawson function**). Considering this in a weak H(0) variational representation ensures that eigenvalues of correspondingly defined convolution integral operator are identical to the zeros of the entire Zeta function (as in a weak L(2) sense every L(2) function g is identical to its Hilbert transform). In quantum theory this goes along with an analysis of signals on R filtered by the Hilbert operator (ZhB). The corresponding analysis with signals on T=R/Z then leads to a replacement of the Gaussian function by the **fractional part function** and its related Hilbert transform **log(2sin)-function**, which is linked to the **cot-function**, building the kernel function of the Hilbert transform for periodic functions. We note that for signals on R the spectrum of the Hilbert transform is (up to a constant) given by the distribution v.p.(1/x), whereby the symbol "v.p." denotes the Cauchy principal value of the integral over R. Its corresponding Fourier series is given by -i*sgn(k) with its relationship to "positive" and "negative" Dirac "functions" and the unit step function Y(x). In a H(-1/2) framework the Dirac "function" concept can be avoided, which enables a generalization to dimensions n>1 without any corresponding additional regularity requirements (the **Dirac** "**function**" is **an element of H(-n/2-e)**, e>0).

Riemann's "workaround" function h(x) do have an obvious linkage to the "**commutator**" concept in quantum theory. In this context the Gaussian function can be characterized as "minimal function" for the Heissenberg uncertainty inequality. Applying the same solution concept as above then leads to an alternative Hilbert operator based representation in H(-1/2), resp. to a H(-1) based definition of the commutator operator with extended domain. The common denominator of the alternatively proposed Hilbert space framework H(-1/2) goes along with the definition of a correspondingly defined "momentum" operator (of order 1) P: H(1/2) --> H(-1/2) defined in a variational form. In the one-dimensional case the Hilbert transform H (in the n>1 case the **Riesz operators R**) is linked to such an operator given by ((Pu,v))=(Hu,v). With respect to quantum theory this indicates an **alternative Schrödinger momentum operator** (where the gradient operator "grad" is basically replaced by the Hilbert transformed gradient, i.e. **P**:=-i***R**(grad) and a corresponding alternative commutator representation Q**P**-**P**Q in a weak H(-1/2) form. We note that the **Riesz operators R** commute with **translations** and **homothesis** and enjoy nice properties relative to **rotations**.

Conceptually, dealing with the isometric mapping Hilbert transform instead of a second order operator in the form x*P(g(x)) (or the commutator (P,Q)) goes along with a few other opportunities. For example, it enables a correspondingly defined variational representation of the Maxwell equations in a vaccum, whereby its solutions do not need any callibration transforms to ensure wave equation character; therefore, the arbitrarily chosen Lorentz condition for the electromagnetic potential (to ensure Lorentz invariance in wave equations) and its corresponding scalar function ((FeR), 7th lecture) can be avoided. At the same point in time it enables alternative concepts in GRT regarding concepts like current (flexible") metrical affinity, affine connexions and local isometric 3D unit spheres dealing with rigid infinitesimal pieces, being replaced by **geometrical manifolds**, enabling isometrical stitching of rigid infinitesimal pieces ((CiI), (ScP)).

The newly proposed "fluid/quantum state" Hilbert space H(-1/2) with its closed orthogonal subspace of H(0) goes also along with a combined usage of L(2) waves governing the H(0) Hilbert space and "orthogonal" wavelets governing the H(-1/2)-H(0) space. The wavelet "reproducing" ("duality") formula provides an additional degree of freedom to apply wavelet analysis with appropriately (problem specific) defined wavelets, where the "*microscope observations*" of ***two*** **wavelet (optics) functions** can be compared with each other (LoA). The prize to be paid is about additional efforts, when re-building the reconstruction wavelet.

In SMEP (Standard Model of Elementary Particles) symmetry plays a key role. Conceptually, the SMEP starts with a set of fermions (e.g. the electron in quantum electrodynamics). If a theory is invariant under transformations by a symmetry group one obtains a conservation law and quantum numbers. Gauge symmetries are local symmetries that act differently at each space-time point. They automatically determine the interaction between particles by introducing bosons that mediate the interaction. U(1) (where probability of the wave function (i.e. the complex unit circle numbers) is conserved) describes the elctromagnetic interaction with 1 boson (photon) and 1 quantum number (charge Q). The group SU(2) of complex, unitary (2x2) matrices with determinant I describes the weak force interaction with 3 bosons (W(+), W(-), Z), while the group SU(3) of complex, unitary (3x3) matrices describes the strong force interaction with 8 gluon bosons.

With respect to the open Millenium **3D non-stationary, non-linear NSE** problem we note that the alternatively proposed **"fluid state" Hilbert space H(-1/2)** with corresponding alternative energy ("**velocity**") space H(1/2) enables a (currently missing) energy inequality based on existing contribution of the non-linear term. In the standard weak NSE representation this term is zero, which is a great thing from a mathematical perspective, avoiding sohisticated estimating techniques, but a doubtful thing from a physical modelling perspective, as this term is the critical one, which jepordized all attempts to extend the 3D problem based on existing results from the 2D case into the

3D case. The corresponding estimates are based on Sobolev embedding theorems; the **Sobolevskii estimate** provides the appropriate estimate given that the "fluid state" space is H(-1/2) in a corresponding weak variational representation.

The electromagnetic interaction has gauge invariance for the probability density and for the Dirac equation. The wave equation for the gauge bosons, i.e. the **generalization** of the **Maxwell equations**, can be derived by forming a gauge-invariant field tensor using generalized derivative. There is a parallel to the definition of the covariant derivative in general relativity. With respect to the above there is an alternative approach indicated, where the fermions are modelled as elements of the Hilbert space H(0), while the complementary closed subspace H(-1/2)-H(0) is a model for the "**interaction particles, bosons**". For gauge symmetries the fundamental equations are symmetric, but e.g. the ground state wave function breaks the symmetry. When a gauge symmetry is broken the gauge bosons are able to acquire an effective mass, even though gauge symmetry does not allow a boson mass in the fundamental equations. Following the above alternative concept the "symmetry state space" is modelled by H(0), while the the ground state wave function is an element of the closed subspace H(-1/2)-H(0) of H(-1/2) (BrK).

A "3D challenge" like the NSE above is also valid, when solving the monochromatic scattering problem on surfaces of arbitrary shape applying electric field integral equations. From (IvV) we recall that the (integral) operators A and A(t): H(-1/2) --> H(1/2) are bounded Fredholm operators with index zero. The underlying framework is still the standard one, as the domains are surfaces, only. An analog approach as above with correspondingly defined surface domain regularity is proposed.

Replacing the affine connexion and the underlying covariant derivative concept by a geometric structure with corresponding inner product puts the spot on the

Thurston conjecture*: The interior of every compact 3-manifold has a canonical decomposition into pieces which have geometric structure (ThW).*

This conjecture asserts that any compact 3-manifold can be cut in a reasonably canonical way into a union of geometric pieces. In fact, the decomposition does exist. The point of the conjecture is that the pieces should all be geometric. There are precisely eight homogeneous spaces (X, G) which are needed for geometric structures on 3-manifolds. The symmetry group SU(2) of quaternions of absolute value one (the model for the weak nuclear force interaction between an electron and a neutrino) is diffeomorph to S3, the unit sphere in R(4). The latter one is one of the eight geometric manifolds above (ScP). We mention the two other relevant geometries, the Euclidean space E3 and the hyperbolic space H3. It might be that our universe is not an either... or ..., but a combined one, where then the "connection" dots would become some physical interpretation. Looking from an Einstein field equation perspective the Ricci tensor is a second order tensor, which is very much linked to the Poincare conjecture, its solution by Perelman and to S3 (AnM). The **geometrodynamics** provides alternative (pseudo) tensor operators to the Weyl tensor related to H3 (CiI). In (CaJ) the concept of a Ricci potential is provided in the context of the Ricci curvature equation with rotational symmetry. The single scalar equation for the Ricci potential is equivalent to the original Ricci system in the rotationally symmetric case when the Ricci candidate is nonsingular. For an overview of the Ricci flow regarding e.g. entropy formula, finite extinction time for solutions on certain 3-manifolds in the context of Prelman's proof of the Poincare conjecture we refer to (KlB), (MoJ).

The single scalar equation for the Ricci potential (CaJ) might be interpreted as the counterpart of the **CLM vorticity equation** as a simple one-dimensional turbulent flow model in the context of the NSE.

The link back to a Hilbert space based theory might be provided by the theory of spaces with an indefinite metric ((DrM), (AzT), (DrM), (VaM)). In case of the L(2) Hilbert space H, this is about a decomposition of H into an orthonal sum of two spaces H1 and H2 with corresponding projection operators P1 and P2 relates to the concepts which appear in the problem of S. L. Sobolev concerning **Hermitean operators** in spaces with **indefinite metric** ((VaM) IV). For x being an element of H this is about a defined "**potential**" p(x):=<<x>>*<<x>> ((VaM) (11.1)) and a corresponding **"grad"** potential operator **W**(x), given by

grad(p(x)):=2**W**(x):=P1(x)-P2(x)   (VaM) (11.4).

The potential criterion p(x)=c>0 defines a manifold, which represents a **hyperboloid in the Hilbert space H** with corresponding hyperbolic and conical regions. The tool set for an appropropriate **generalization** of the above "grad" definition is about the (homogeneous, not alway non-linear in h) **Gateaux differential** (or weak differential) **VF**(x,h) of a functional F at a point x in the direction h ((VaM) §3)). The appropriate weak inner product might be the inner product of the "velocity" space H(1/2). We note the Sobolev embedding theorem, i.e. H(k) is a sub-space of C(0) (continuous functions) for k>n/2, i.e. there is no concept of "continuous velocity/momentum" in the proposed Hilbert space framework, i.e. there is no Frechet differential existing ((VaM) 3.3). This refers to one of the several proposals, which have been made to drop some of the common sense notions about the universe ((KaM) 1.1), which is about continuity, i.e. space-time must be granular. The size of these grains would provide a natural cutoff for the Feynman integrals, allowing to have a finite S-matrix.

A selfadjoint operator B defined on all of the Hilbert space H is bounded. Thus, the operator B induces a decomposition of H into the direct sum of the subspaces, and therefore generates related hyperboloids ((VaM) 11.2). Following the investigations of Pontrjagin and Iohvidov on linear operators in a Hilbert space with an indefinite inner product, M. G. Krein proved the Pontrjagin-Iohvidov-Krein theorem (FaK).

In an universe model with appropriately connected geometric manifolds the corresponding symmetries breakdowns at those "connection dots" would govern corresponding different conservation laws in both of the two connected manifolds. The Noether theorem provides the corresponding mathematical concept (symmetry --> conservation laws; energy conservation in GT, symmetries in particle physics, global and gauge symmetries, exact and broken). Those symmetries are associated with "non-observables". Currently applied symmetries are described by finite- (rotation group, Lorentz group, ...) and by infinite-dimensional (gauged U(1), gauged SU(3), diffeomorphisms of GR, general coordinate invariance...) Lie groups.

A manifold geometry is defined as a pair (X,G), where X is a manifold and G acts transitively on X with compact point stabilisers (ScP). Related to the key tool "Hilbert transform" resp. "conjugate functions" of this page we recall from (ScP), that Kulkarni (unpublished) has carried out a finer classification in which one considers pairs (G,H) where G is a Lie group, H is a compact subgroup and G/H is a simple connected 3-manifold and pairs (G1,H1) and (G2,H2) are equivalent if there is an isomorphism G1 --> G2 sending H1 to a conjugate of H2. Thus for example, the geometry S3 arises from three distinct such pairs, (S3,e), (U(2),SO(2)), (SO(4),SO(3)). Another example is given by the Bianchi classification consisting of all simply connected 3-dimensional Lie groups up to an isomorphism.

2. An integrated substance & field least action functional framework for elementary particles and gravity "forces" phenomena

The central mathematical concepts of the GRT are differentiable manifolds, affine connexions with the underlying covariant derivative definition on corrresponding tangential (linear) vector spaces. Already the "differentiability" condition is w/o any physical justification. The only "affine" connexion concept and its corresponding locally defined metrical space framework jeopardizes a truly infinitesimal geometry, which is compatible with the Hilbert space framework of the quantum theory and the proposed distributional Hilbert space concept in (BrK). In sync with the above we propose a generalized Gateaux differential operator:

let H(1/2) = H(1) + H(*) denote the orthogonal decomposition of the alternatively proposed "energy/momentum/velocity" Hilbert space, whereby H(1) denotes the (compactly embedded) standard energy space with its inner product, the Dirichlet integral; "lim" denotes the limes for t --> 0 for real t. Then for x,y ex H(1/2) the operator VF(x,y) is defined by VF(x,y):=lim((F(x+t*y)-F(x))/t), whereby the limes is understood in a weak H(-1/2) sense. The operator is homogeneous in y; however, it is not always a linear operator in y ((VaM) 3.1).

The main tools used in geometrical theory of gravitation are tensor fields defined on a Lorentzian manifold representing space-time. A Lorentz manifold L is likewise equipped with a metric tensor g, which is a nondegerated symmetric bilinear form on the tangential space at each point p of L. The Minkowski metric is the metric tensor of the (flat space-time) Minkowski space.

The least action principle can refer to the family of variational principles. The most popular among these is Hamilton's principle of least action. It states that the action is stationary under all path variations q(t) that vanishes at the end points of the path. It does not not strictly imply a minimization of the action.

The **least action principle** can be also seen as THE fundamental principle to develop laws of nature in strong alignment with Kant's philosophy: ((KnA), p. 55, p. 56): (translated*) "the least action principle in his most modern general public is a* ***maxime of Kant's reflective judgment***. ... *Offenbar haben wir beim Energieprinzip eine typische Entwicklung vor uns: wenn das Prinzip der reflektierenden Urteilskraft mit einer seiner Maximen vollen Erfolg gehabt hat, rückt sein Ergebnis aus dem Reich der Vernunft im Kantischen Sinne, zu welchem die reflektierende Urteilskraft gehört, in die Sphäre des Verstandes herab und wird zum allgemeinen Naturgesetz (****law of nature****)*".

The **Einstein-Hilbert action functional** W(g) is about the **scalar curvature** S=scal (which is the Ricci scalar of the Ricci tensor "Ric") applied to the metric tensor g. It is the simplest curvature invariant of a Riemannian manifold. The scalar curvature is the Lagrangian density for the Einstein-Hilbert action. The stationary metrics are known as Einstein metrics. The scalar curvature is defined as the trace of the Ricci tensor. We note that the trace-free Ricci tensor for space-time dimension n=4 is given by Z(g):=Ric(g)-(1/4)*S(g)*g, and that Z vanishes identically if and only if Ric = l*g for some constant l. In physics, this equation states that the manifold (M,g) is a solution of Einstein's vacuum field equations with cosmological constant. We further note, that the Ricci tensor corresponds to the Laplacian operator multiplied by the factor (-1/2) plus lower order terms.

The Einstein-Hilbert action functional leads to the Einstein field equations with the Einstein tensor G := Ric(g)-(1/2)*S(g)*g, whereby the negative Einstein tensor -G is the L(2) gradient of the Einstein-Hilbert functional W(g). Its counterpart in elasticity theory is given by the **principle of Castigliano** (for elastic bodies), which is about the minimization of the potential energy. It is a weak form representation of the corresponding classical boundary value problem -grad(S(**u**))=f and boundary conditions (!), whereby, in this case, S denotes the stress tensor ((VeW) (4.127), (4.128)), i.e. what's missing in the Einstein-Hilbert action representation are appropriate "boundary" conditions. This links back to the "origin of inertia in the Einstein geometrodynamics" to develop a modified well defined Einstein-Hilbert functional in sync with appropriate Hilbert scale (CiI). In (HoA) a generalized concept of Minkowski space is provided embedded in a semi-indefinite-inner-product space using the concept of a new product, that contains the classical cases as special ones. It is proposed as alternative (integration) concept for the 3 geometries of the (special relativity) universe, which are the Minkowski, the de Sitter and the anti-de Sitter space with corresponding zero, positive and negative curvature.

The Einstein-Hilbert functional is an *invariant* integral, which is a must to describe the field-action of graviation ((WeH), §28). From a physical perspective a field-action term should be based on a scalar density G, which is composed of the potentials g(i,k) and of the field-components of the gravitation field (which are the first derivatives of the g(i,k,), i.e., g(i,k);r): "*it would seem to us that only under such circumstances do we obtain differential equations of order not higher than the second for our gravitation laws .... Unfortunately a* ***scalar density G****, of the type we wish, does not exist at all; for we can make all g(i,k);r vanish at any given point choosing the appropriate co-ordinate system. Yet, the scalar R, the curvature defined by Riemann, has made us familiar with an invariant which involves the second derivatives of the g(i,k)'s only lineary. ... In consequence of this linearity we may use the invariant integral (the Einstein-Hilbert functional) to get the derivatives of the second order by partial integration. ... We then get a sum of a truly field-action functional (*with a scalar density G*) plus a divergence integral, that is an integral whose integrand is of the form div(****w****). Hence for the corresponding variations of theh potential functions g(i,k) the variations of both funtionals are identical; therefore the replacement of the physically required scalar density G by the integrand of the W(g) is justified (as the essential feature of the Hamilton's principle is fulfilled with W(g)).*" This is where a alternative field-action functional of gravitation in a alternative framework (as proposed above) can be defined, based on a **"*scalar density*" function** in a "**Plemelj**" (Stieltjes integral) sense.

The electromagnetic field is built up from the co-efficients of an invariant *linear* differential form. The potential of the gravitational field is made up of the co-efficients of an invariant quadratic differential form. Replacing the Newtonian law of attraction by the Einstein theory is about discovering the invariant law of gravitation, according to which matter determines the components of the graviation fields. The topic of the chapter above is about the **substance-action** and the **field-action** of electricity and gravitation in the context of the least action principle. The substance-action is based on the mathematical concept "density", while the field action is based on the mathematical concept "potential (function)". The substance-action related "**tensor density**" of electricity can be easily extented to the substance-action related "tensor density" of gravitation ((WeH) §28). A corresponding field-action of gravitation based on an invariant integral *and* an approporate potential "scalar density" is not possible from a mathematical perspective, as by choosing the appropriate co-ordinates the field components of the gravitational field vanish. The alternatively proposed aproach of this page can be summarized as follows:

- replacing of the mathematical "*density*" concept by Plemelj's "mass element" concept, which goes along with an alternative (more general) "*potential*" function concept

- replacing the manifold concept by a (semi) Hilbert space-based concept, where a non-linear invariant integral functional F(V(g)) is defined by a distributional (semi-) inner product, which is equivalent to a corresponding functional F(R(g)) of a related inner product (where R denotes the Riesz operators (which commute with translations & homothesis having nice properties relative to rotations)) plus a (non-linear) compact disturbance term; the concept enables variational methods of nonlinear operators based on Stieltjes and curvilinear integrals (VaM).

The Yang-Mills functional is of similar structure than the Maxwell functional regarding the underlying constant fundamental tensor. The field has the property of being self-interacting and equations of motions that one obtains are said to be semilinear, as nonlinearities are both with and without derivatives. The **YME mass gap problem** is about the **energy gap for the vacuum state**. Therefore, the above proposed model alignments for the "electricity & gravity forces" phenomena covers also the cases of the "weak & strong nuclear forces" phenomena.

To merge two inconstent theories requires changes on both sides. In the above case this is about a newly proposed common "mass/substance element" concept, alternatively to the today's "mass density" concept, while, at the same time, the linear algebra tensor tool (e.g. a "density" tensor) describing classical PDE systems is replaced by non-linear operator equations defined by weak (variational) functional systems. Those (weak) equations provides the mathematical model of physical phenomena, while its correspondin classical PDE systems (requiring purely mathematical additional regularity assumptions) are interpreted as approximation solutions, only.

3. Geometrodynamics, distortion-free, progressing Maxwell and Einstein waves and space-time matter

As a shortcut reference to geometrodynamics is given by (WhJ). For a review of discoveries in the nonlinear dynamics of curved spacetime, we refer to ((ScM). An introduction to the foundations and tests of gravitation and geometrodynamics or the meaning and origin of inertia in Einstein theory is provided in (CiI).

In ((CiI) 4.6) the Gödel model universe is discussed, which is a four-dimensional model universe, homogeneous both in space and time, which admits the whole four-dimensional simply transitive group of isometries, in other words, a space-time that admits all four "simple translations" as independent Killing vectors. As the Gödel model universe is homogeneous both in space and time it is stationary. In other words, in this model the cosmological fluid is characterized by zero expansion and zero shear. Thus the Gödel model runs into difficulty with the expansion of the universe.

The initial-value problem and the interpretation of the origin of inertia in geometrodynamics is considered in ((CiI) 5.1, 5.2):

"*The specification of the relevant features of a three-geometry and its time rate change on a closed (compact and without boundary manifolds), initial value, space-like hypersurface, together with the energy density and density of energy flow (conformal) on that hypersurface and together with the expansion of the equation of state of mass-energy, determines the entire space-time geometry, the local inertial frames, and hence the inertial properties of every test particle and every field everywhere and for all time.*"

The related clarifcations regarding the distortion tensor or gravitomagnetic field is provided in ((CiI) §5.2.6, § 5.2.7).

The Laplacian equation for the gravitomagnetic vector potential W, in terms of the current J of mass-energy is discussed in ((CiI) 5.3). The Neumann problem and its related integral equations with double layer potential leads to the Prandtl operator, defining a well posed integral equation in case of domain H(1/2) with range H(-1/2) ((LiI) theorem 4.3.2).

The Prandtl operator with appropriate domain ((LiL) theorem 4.3.2) is proposed to be applied defining an adequate (distributional) Hilbert space framework for the geometrodynamics (GMD) (gravitation & inertia and 3 manifolds geometries (ScP), requiring an appropriate definition of a corresponding inner product. This proposed inner product (in opposite to the standard "*exterior*" product) is in line with the idea of (BrK), defining an alternative Hilbert space framework for an alternative new ground state energy model (for the harmonic quantum oscillator model). In essence it is about a inner product (and corresponding norm = metric) of "Plemelj's mass elements" (represented as 1-forms (i.e. differentials)):

$$((((du,dv)))):=(((u,v))):=((P(u),v))$$

whereby (((*,*))) defines the H(-1) inner product and ((*,*)) defines the H(-1/2) inner product of the corresponding Hilbert scales building on the eigen-pair solutions of the Prandtl operator equation with domain H(1/2).

The proposed alternative Hilbert space based framework provides also a "variational wave equation/ function" based approach of the "evolution of geometric structures on 3-manifolds" in the context of Thurston's "**geometrization conjecture**" and its underlying **Poincare conjecture** (which have been established by **Perelman**), where the **Ricci**

flows play a central conceptual solution element to build "nice behavior" metrics in manifolds.

"T*he hypothesis that the* ***universe is infinite*** *and* ***Euclidean at infinity****, is, from a relativistic point of view, a complicated hypothesis. In the language of the general theory of relativity it demands that the* ***Riemann tensor*** *of the fourth rank* ***shall vanish at infinity****, which furnishes twenty independent conditions, while* ***only ten curvarture components*** *enter the laws of the* ***gravitational field****. It is certainly unsatisfactory to postulate such far-reaching limitation without any physical basis for it.*

If we think these ideas consistently through to the end we must expect ***the whole inertia****, that is, the whole* ***g(i,k)-field****,* ***to be determined by the matter of the universe****, and* ***not*** *mainly by the* ***boundary conditions at infinity****.*

The possibility seems to be particularly satisfying that the universe is spatially bounded and thus, in accordance with our assumption of the constancy of the mass-energy density, is of constant curvature, being either spherical or elliptical; for then the boundary conditions at infinity which are so inconvenient from the standpoint of the general theory of relativity, may be replaced by the much more natural conditions for a closed surface" ((CiI) 5.2.1)

The wave equation can be derived from the Maxwell equations by applying the rot-operator. It results into the "light" phenomenon. A similar tranformation is not possible for Einstein equations, which results into the "gravitation" phnomenon. The "approximation" approach is about the split g(i,k)=m(i,k)+h(i,k), where m(i,k) denotes the flat Minkowski metric. The perturbance term h(i,k) admits a retarded (only) potential representation, representing a gravitational perturbance propagating at the speed of light ((CiI) 2.10). An alternative splitting with defined distortion tensor enabling an analogue approach with electrodynamics is provided in ((CiI) 5.2.7).

In ((CiI) (2.7.10)) an „*energy-momentum pseudotensor for the gravity field*" is introduced representing the energy and momentum of the gravitation field. Then, using the corresponding *"**effective energy-momentum pseudotensor for matter, fields and gravity field***", in analogy with special relativity and electromagnetism, the conserved quantities on an asymptotically flat spacelike hypersurface are defined by the sum of four-momentum, energy and angular momentum operators (2.7.19-21). Following an analogue approach, which lead to the modified Maxwell equation (as proposed in the above paper), leads to an alternative *effective energy-momentum tensor for matter, fields and gravity field*". As the Einstein (gravity) tensor is derived from the condition of a divergence-free energy-momentum tensor, this results to an alternative Einstein tensor. The additional term of this alternative Einstein tensor could be interpreted as "cosmologic term", not to ensure a static state of the universe (which is not the case due to Hubbles observations), but to model the "vacuum energy" properly. This then would also be in sync with the physical interpretation of the corresponding term in the modified Maxwell equations with its underlying split of divergence-free and rotation-free tensors. At the same point in time the approach avoids the *affine connexion* concept and the "*differentiable*" manifolds regularity requirement, which is w/o any physical justification.

There are eight 3-dimensional geometries in the context of "nice" metrics. The nicest metrics are those with a constant curvature, but there are other ones. Their classification in dimension three is due to Thurston (ScP).

In (GrJ) philosophical aspects of the geometrodynamics are considered. We quote from the cover letter summary:

"*The central conceptual idea of the* ***contemporary theory of general relativity*** *– or* *geometrodynamics – is the **identification of matter with the structure of space-time**. No identities foreign to space-time, like masses, charges, or independent fields are needed, and physics thus becomes identical with the geometry of space-time. This idea implies a philosophical description of the universe that is monistic and organic, characterized by an all-encompassing interdependence of events. …. The Newtonian independence and distinctness of objects is at the polar extreme from their Einsteinian interdependence and continuity. …. He (the author) then presents the remarkable recent developments in geometrodynamics which allow the program of identifying matter with space-time to be carried further than even Einstein suspected possible. The surprising discovery that electromagnetism can be incorporated into geometrodynamics without modifying Einstein's original equations appears to be formally correct, but reliance on multiply connected topologies ("**wormholes**") to represent charge raises various unresolved questions. Graves concludes that the present language of physics, like that of every-day life, is based on concepts of independence and separation, and that a wholly new language may needed to describe the world in terms of geometrodynamics, in which space-time appears as the only substance, with curvatures as its attributes, and in which objects have no absolute individuality, distinctness, or location.*"

The above *questions* concerning singularities and non-geometric manifolds can be revisited based on the above alternative conceptual framework; the corresponding physical interpretation of the geometrodynamics are in line with Schrödinger's vision (resp. critique about the common handicap of all western philosophy baseline assumptions, propagating instead a purely monoism) of a truly quantum field theory (see also www.quantum-gravitation.de).

In (CoR) there is a **conjecture** formulated, that **distortion-free families of progressing, spherical waves of higher order** exist if and only if the **Huyghens' principle** is valid, and that families of spherical, progressing waves only exist for space-time dimension n=2 and n=4 ((CoR) VI, §10.2, 10.3). In combination with Hadamard conjecture (that the wave equations for even space-time dimension are the only partial differential equations, where the Huyghens' principle is valid) this would lead to an essential characterization of the four-dimension space-time space with its underlying Maxwell field theory.

With respect to the geometrodynamics (gravitation and inertia) we note that ..

1. .. Huyghens' principle is valid under the same conditions for both, the initial value problem of the wave equation and the corresponding radiation problem. For each t>0 the latter one is defined by a certain sphere integral limit regarding of the normal derivative defining the intensity of the radiation as function of the time variable. Spherical waves are defined in that way, that the family of its corresponding characteristic surfaces builds characteristics conoids, those tips lie all on a time-like curve ((CoR) VI, §10.1).

2. *.. in order to avoid the problem of existence of closed time-like curves and the problem of special non-compactness, Gödel proposed a rotating cosmological model that have no closed time-like curves and that expand but are spatially homogeneous and compact* (CiI), 4.6, 4.7)

3. *.. there is an alternative postulate that space geometry shall be asymptotically flat with two problems of principle, (1) it imposes "flatness from on high", (2) "the quantum fluctuations rule the geometry of space in the small". No natural escape has ever presented itself from these two difficulties of principle excepts to say that space in the large must be compact. No one will deny that space-time approaches flatness well out from many a localized center of attraction. However, nothing, anywhere, in any finding of astrophysics of our day makes it unattractive to treat every such nearly flat region, or even totally flat region, "not as infinite, but as part of a closed universe". … The role of spatial closure in the context of finding the magnetic field associated with a stationary system of electric currents lead to an additionally to be added magnetic field, that is free*

of curl and divergence (and which therefore goes on and on to infinite with its twisty, wavy lines of force) to the well-known obvious solution to obtain another (unique) solution. Transferring this concept to geometrodynamics is tame, when the S(3) topology is supplemented by one or more wormholes. Then the solution is not unique until one restores uniqueness by specifying the flux through each wormhole (CiI) (5.2.1)).

Related to topic 1 above we note that spherical waves are only relative distortion-free and progressing due to the special role of the time-like curves.

The singularities of wormholes are the main challenges of current status of geometrodynamic (topic 3 above). We propose Plemelj's alternative "mass element", "flux" and "flux strength" concept to specify the inertia condition for the corresponding radiation conditions (in analogy to the wave equation radiation condition (CoR). It is based on an alternative "normal derivative" concept. Its definition requires only information from the surface. The corresponding field equations are defined in a (weak) variational representation based on a H(-1/2) "space-time matter fluid/particle". It avoid the Dirac "function" concept, which is a "singularity governing function". It avoid the concept of "continuity" resp. "differentiable continuity", requiring regularity conditions to enable the Sobolev embedding theorem (H(k) sub-space of C(0), if k>n/2).

We mention that the existing electromagnetic phenomena on earth are the result of plasma physics phenomena underneath the earth crust. Those "activities" are all triggered by gravitation "forces".

The above (distributional) Hilbert space based alternative geometrodynamic modelling framework provides an alternative approach to **Penrose**'s "**cycles of time**" concept of a "**conformal cyclic cosmology**", addressing e.g. the "collapsing of matter" of an over-massive star to a black hole problem (PeR) and "the problem of time" (AnE).

"*What characterizes the **loop quantum theory** (LQT) is the choice of a different algebra of basic field functions: a noncanonical algebra based on the holonomics of the gravitational connections ((RoC) 1.2.1). The holonomy (or the "Wilson loop") is the matrix of the parallel transport along a closed curve. ... In LQT, the holonomy becomes a quantum operator that creates "loop states" (to overcome the issue of current dynamics model of coupled gravity + matter system, simply defined by adding the terms defining the matter dynamics to the gravitational relativistic hamiltonian ((RoC) 7.3)). ... **Spacetime** itself is **formed** by **loop-like states**. Therefore the position of a loop state is relevant only with respect to other loops, and not with respect to the background. ... The state space of the theory is a separable Hilbert space spanned by loop states, admitting an orthogonal basis of spin network states, which are formed by finite linear combinations of loop states, and are defined precisely as the spin network states of a lattice **Yang-Mills theory**.*" The proposed distributional quantum state H(-1/2) above admits and requires infinite linear combinations of those "loop states" (which we call "quantum fluid" state), i.e. overcomes the **current challenge of LQT** defining the scalar product of the spin network state Hilbert space ((RoC) 7.2.3). The physical space is a quantum superposition of "***spin networks***" in LQT corresponds to an orthogonal projection of H(-1/2) onto H(0). This othogonal projection can be interpreted as a general model for a "**spontaneous symmetry break down**".

4. Plasma physics, Maxwell equations & non-linear Landau damping

Plasma is the fourth state of matter, where from general relativity and quantum theory it is known that all of them are fakes resp. interim specific mathematical model items. An adequate model needs to take into account the *axiom of (quantum) state* (physical states are described by vectors of a separable Hilbert space ***H***) and the *axiom of observables* (each physical observable ***A*** is represented as a linear Hermitian operator of the state Hilbert space). The corresponding mathematical model and its solutions are governed by the Heisenberg uncertainty inequality. As the observable space needs to support statistical analysis the Hilbert space, this Hilbert space needs to be at least a subspace of ***H***. At the same point in time, if plasma is considered as sufficiently collisional, then it can be well-described by fluid-mechanical equations. There is a hierarchy of such hydrodynamic models, where the magnetic field lines (or magneto-vortex lines) at the limit of infinite conductivity is "frozen-in" to the plasma. The "mother of all hydrodynamic models is the *continuity equation* treating observations with macroscopic character, where fluids and gases are considered as continua. The corresponding infinitesimal volume "element" is a volume, which is small compared to the considered overall (volume) space, and large compared to the distances of the molecules. The displacement of such a volume (a fluid particle) then is a not a displacement of a molecule, but the whole volume element containing multiple molecules, whereby in hydrodynamics this fluid is interpreted as a mathematical point.

The common distributional Hilbert space framework is also proposed for a proof of the Landau damping alternatively to the approach from C. Villani. Our approach basically replaces an analysis of the classical (strong) partial differential (Vlasov) equation (PDE) in a corresponding Banach space framework by a quantum field theory adequate (weak) variational representation of the concerned PDE system. This goes along with a corresponding replacement of the "hybrid" and "gliding" analytical norms (taking into account the transfer of regularity to small velocity scales) by problem adequate Hilbert space norms H(-1/2) resp. H(1/2). The latter ones enable a "fermions quantum state" Hilbert space H(0), which is dense in H(-1/2) with respect to the H(-1/2) norm, and its related (orthogonal) "bosons quantum state" Hilbert space H(-1/2)-H(0), which is a closed subspace of H(-1/2).

With respect to the above alternative RH theory we recall that the Zeta function Z(t) (s=1/2+it) on the critical line is an element of the Hilbert space H(-1). Its related weak variational representation with respect to the H(0) test space defines a corresponding "weak" Zeta function representation, which is an element of the (more regular, quantum state) Hilbert space H(-1/2).

We propose modified Maxwell equations with correspondingly extended domains according to the above. This model is proposed as alternative to SMEP, i.e. the modified Maxwell equation are proposed to be a "Non-standard Model of Elementary Particles (NMEP)", i.e. an alternative to the Yang-Mills (field) equations. The conceptual approach is also applicable for the Einstein field equations. Mathematical speaking this is about potential functions built on correponding "density" functions. The source density is the most prominent one. Physical speaking the source is the root cause of the corresponding source field. Another example is the invertebrate density (=rotation) with its corresponding rotation field. The Poincare lemma in a 3-D framework states that source fields are rotation-free and rotation fields are source-free. The physical interpretation of the rotation field in the modified Maxwell equations is about rotating "mass elements w/o mass" (in the sense of Plemelj) with corresponding potential function. In a certain sense this concept can be seen as a generalization of the Helmholtz decomposition (which is about a representation of a vector field as a sum of an irrotational (curl-free) and a solenoidal (divergence-free) vector field): it is derived applying the delta "function" concept. In the context of the proposed distributional Hilbert space framework, the Dirac function concept (where the regularity of those "function" depends from the space dimension) is replaced by the quantum state Hilbert space H(-1/2). The solution u (ex H(1/2)) of the Helmholtz equation in terms of the double layer potential is provided in ((LiI), 7.3.4). From the Sobolev embedding theorem it follows, that for any space dimension n>0 the modified Helmholtz equation is valid for not continuous vector fields.

The **Boltzmann** equation is a (non-linear) integro-differential equation which forms the basis for the kinetic theory of gases. This not only covers classical gases, but also electron /neutron /photon transport in solids & plasmas / in nuclear reactors / in super-fluids and radiative transfer in planetary and stellar atmospheres. The Boltzmann equation is derived from the Liouville equation for a gas of rigid spheres, without the assumption of "molecular chaos"; the basic properties of the Boltzmann equation are then expounded and the idea of model equations introduced. Related equations are e.g. the Boltzmann equations for polyatomic gases, mixtures, neutrons, radiative transfer as well as the Fokker-Planck (or Landau) and Vlasov equations. The treatment of corresponding boundary conditions leads to the discussion of the phenomena of gas-surface interactions and the related role played by proof of the Boltzmann H-theorem.

The **Landau** equation (a model describing time evolution of the distribution function of plasma consisting of charged particles with long-range interaction) is about the Boltzmann equation with a corresponding **Boltzmann collision operator** where almost all collisions are grazing. The mathematical tool set is about Fourier multiplier representations with Oseen kernels (LiP), Laplace and Fourier analysis techniques (e.g. [LeN]) and scattering problem analysis techniques based on Garding type (energy norm) inequalities (like the Korn inequality). Its solutions enjoy a rather striking compactness property, which is main result of P. **Lions** ((LiP) (LiP1)).

The **Landau damping** (physical, observed) phenomenon is about "*wave damping w/o energy dissipation by collisions in plasma*", because electrons are faster or slower than the wave and a Maxwellian distribution has a higher number of slower than faster electrons as the wave. As a consequence, there are more particles taking energy from the wave than vice versa, while the wave is damped. The (kinetic) Vlasov equation is collisions-less.

In fluid description of plasmas (**MHD**) one does not consider velocity distributions. It is about number density, flow velocity and pressure. This is about moment or fluid equations (as NSE and **Boltzmann**/**Landau** equations). The corresponding situation of the fluid flux of an incompressible viscous fluid leads to the Navier-Stokes equations. They are derived from continuum theory of non-polar fluids with three kinds of balance laws: (1) conservation of mass, (2) balance of linear momentum, (3) balance of angular momentum.

The **NSE** are derived from the (Cauchy) stress tensor (resp. the shear viscosity tensor) leading to liquid pressure force. In electrodynamics & kinetic plasma physics the linear resp. the angular momentum laws are linked to the electrostatic (mass "particles", collision, static, quantum mechanics, displacement related; "fermions") Coulomb potential resp. to the magnetic (mass-less "particles", collision-less, dynamic, quantum dynamics, rotation related; "bosons") Lorentz potential.

When one wants to treat the time-harmonic Maxwell equations with variational methods, one has to face the problem that the natural bilinear form is not coercive on the whole Sobolev space. On can, however, make it coercive by adding a certain bilinear form on the boundary of the domain (vanishing on a subspace of H(1)), which causes a change in the natural boundary conditions.

The mathematical tool to distinguish between unperturbed cold and hot plasma is about the Debye length and Debye sphere. The corresponding interaction (Coulomb) potential of the **non-linear Landau damping model** is based on the (Poisson) potential equation with corresponding boundary conditions. A combined electro-magnetic plasma field model needs to enable “interaction” of cold and hot plasma “particles”, which indicates Neumann problem boundary conditions.

(BrK) Braun K., A distributional Hilbert space framework to prove the Landau damping phenomenon

As a shortcut reference to the underlying mathematical principles of classical fluid mechanics we refer to(SeJ).

Earliest examples of complementary variational principles are provided by the energy principle of Dirichlet in the theory of electrostatics, together with the Thomson principles of complementary energy. As a short cut reference in the context of the considered Maxwell equations we refer to (ShM1).

A central concept of the proposed solution Hilbert space frame is the alternative normal derivative concept of Plemelj. It is built for the logarithimc potential case based on the Cauchy-Riemann differential equations with its underlying concept of conjugate harmonic functions. Its generalization to several variables is provided in the paper below. It is based on the equivalence to the statement that a vector u is the gradient of a harmonic function H, that is u=gradH. Studying other systems than this, which are also in a natural sense generalizations of the Cauchy-Riemann differential equations, leads to representations of the rotation group (StE).

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