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INTEGRAL AND INTEGRO-DIFFERENTIAL EQUATIONS

Generalized Functions on Hilbert Spaces, Singular Integral Equations, and Problems of Aerodynamics and Electrodynamics

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INTRODUCTION

External flow suction devices are used in aerodynamics to increase the lift of an airfoil in an ideal incompressible flow [1]. When solving this problem, the airfoil is modeled by a vortical layer [2], and the external flow suction device is modeled by a sink [3]. The desired velocity field around the profile should have the following property. In a neighborhood of the sink, the velocity field should have a sink-type singularity on the external side of the airfoil (where the sink lies), and it should be smooth on the opposite side. Naturally, if the airfoil satisfies the no-flow condition on the sink side, the sink point itself is excluded from this condition. The airfoil flow problem is thereby reduced to the solution of a singular integral equation on the profile contour with right-hand side undefined at the sink point. It follows from the analysis of tangential components of the velocity field in a neighborhood of the sink on the airfoil side where the velocity field is smooth that the solution of the resulting singular integral equation should be sought in the class of functions that have a singularity of the type of $1/x$ at the sink point [1]. In the numerical solution of this singular integral equation by the discrete vortex method, grid points are chosen so as to ensure that the sink point is one of them, and when replacing the singular integral equation by a system of linear algebraic equations, the equation corresponding to the sink point is omitted [1]. The omitted equation is replaced by another equation, which is derived from some physical considerations. This proves to be inconvenient, since, when using several external flow suction devices, one always has to decide how to fill in the missing equations. Hence it was suggested in [4] to satisfy the no-flow condition not on the side of the airfoil, where the sink lies, but on the opposite side, where the velocity field is smooth. This approach results in the appearance of a delta function supported at the sink point on the right-hand side of the corresponding singular integral equation. Now the singular integral equation should be treated as a pseudodifferential equation in the class of distributions. A version of such interpretation for a singular integral equation with a Hilbert kernel in the class of periodic distributions was presented in [5]. In this case, periodic distributions were treated as a subset of distributions on the entire real line. But this is inconvenient for singular integral equations on an interval in the class of distributions. However, it became simpler to use the discrete vortex method for the numerical solution of singular integral equations for the case in which the right-hand side contains a delta function. The method acquired the same classical form [1] as for singular integral equations in the class of absolutely integrable functions, the delta function being replaced by the corresponding step function [6].

Note also that the solution of the problem on the computation of the input resistance of a thin wire antenna [7] powered by a current source leads to the analysis of a hypersingular integral equation on an interval, where the right-hand side contains a function with a singularity of the type of $1/x$ inside the solution domain. The solution of the corresponding characteristic equation was performed with the use of a spectral relation for a hypersingular integral operator on an interval, which provided a solution with a jump discontinuity at the point of the singularity of the right-hand side. In general, this solution should be considered in the class of distributions. This extension of the classical interpretation of singular and hypersingular integral equations and the corresponding operators resulted in studies in the theory of distributions. These studies are presented in this paper.

1. A VERSION OF GENERALIZED FUNCTIONS ON HILBERT SPACES

Consider an infinite-dimensional real Hilbert space H. (For a complex space, the forthcoming considerations remain valid with the corresponding modifications.) We denote the inner product of vectors $f, g \in H$ by (f, g). Let a vector system $\{\psi_1, \psi_2, \dots, \psi_n, \dots\}$ be an orthonormal basis in the space H, i.e., satisfy the conditions

$$(\psi_i, \psi_j) = \delta_{i,j}, \tag{1.1}$$

where $\delta_{i,j}$ is the Kronecker delta. Then each vector f in H can be represented in the form

$$f = \sum_{k=1}^{\infty} a_k \psi_k, \tag{1.2}$$

where the coefficients a_k satisfy the relation $\sum_{k=1}^{\infty} a_k^2 < \infty$. Moreover, the norm of the vector f is given by the formula

$$\|f\| = \sqrt{(f, f)} = \left(\sum_{k=1}^{\infty} a_k^2 \right)^{1/2}. \tag{1.3}$$

Now, in the space H, we define a test set S of vectors.

Definition 1. A vector f in the space H belongs to the test set S if its expansion in the basis has finitely many nonzero coefficients, i.e., if for the expansion (1.2), there exists a positive integer K such that

$$a_k = 0, \qquad k > K. \tag{1.4}$$

Note that the set $S = S(H)$ is linear and everywhere dense in the space H but is not closed in H. Indeed, each vector f in H with infinitely many nonzero coefficients in the representation (1.2) is the limit of the sequence of vectors

$$f_n = \sum_{k=1}^{n} a_k \psi_k \tag{1.5}$$

in the metric of the space H. Therefore, in the set S, we introduce a different notion of convergence of vectors.

Definition 2. We say that a sequence of vectors $f^{(1)}, f^{(2)}, \dots, f^{(n)}, \dots$ in the set S converges to a vector f in the finite-dimensional sense if there exists a common index K such that

$$a_k^{(n)} = 0, \qquad k > K, \qquad n = 1, 2, \dots, \tag{1.6}$$

and the condition

$$\lim_{n \to \infty} \left| a_k^{(n)} - a_k \right| = 0, \qquad k = 1, 2, \dots, \tag{1.7}$$

is satisfied for each $n = 1, 2, \dots$ We briefly write this in the form

$$f^{(n)} \to f(S). \tag{1.8}$$

In other words, a sequence of vectors $f^{(1)}, f^{(2)}, \dots, f^{(n)}, \dots$ in S converges to a vector f in H if all of them lie in a same finite-dimensional subspace of H and relation (1.7) is satisfied. Therefore, the vector f lies in the same finite-dimensional subspace and hence belongs to the set S. It follows that the finite-dimensional convergence in S provides that S is closed. Now the set S with the above-introduced finite-dimensional convergence is referred to as the space S of test vectors. One can also say that the finite-dimensional coordinate convergence is introduced in S.

We also introduce the notion of a distribution.

Definition 3. A *distribution* is an arbitrary continuous linear functional F on the space S of test vectors. The value of the functional F on a test vector f is denoted by $F(f)$ or (F, f). The continuity of a functional is understood as follows. A functional F is said to be continuous on S if the finite-dimensional convergence of a sequence $f^{(1)}, f^{(2)}, \ldots, f^{(n)}, \ldots$ of vectors in the space S to a vector f in the same space implies that

$$\lim_{f^{(n)}\to f} \left(F, f^{(n)}\right) = (F, f). \tag{1.9}$$

The set of all distributions is denoted by $S' = S'(H)$. The set S' is linear if the linear combination $\lambda F + \mu G$ of distributions F and G is defined as the functional acting by the rule

$$(\lambda F + \mu G, f) = \lambda(F, f) + \mu(G, f), \qquad f \in S. \tag{1.10}$$

Following [8], one can show that the functional $\lambda F + \mu G$ is linear and continuous on S, i.e., belongs to the set S'.

Note that the coordinate convergence in S implies the following assertion.

Theorem 1. *An arbitrary linear functional on S is continuous on S in the sense of Definition* 3.

Proof. Indeed, let a sequence $f^{(1)}, f^{(2)}, \ldots, f^{(n)}, \ldots$ of vectors in the space S converge in the finite-dimensional sense to a vector f in S. It follows that, for the coefficients $\{a_k^{(n)}\}$ in the expansion of the vectors $f^{(n)}$, there exists a K such that $a_k^{(n)} = 0$, $k > K$, $n = 1, 2, \ldots$ Therefore, if F is a linear functional on S, then

$$\begin{aligned}\lim_{f^{(n)}\to f} \left(F, f^{(n)}\right) &= \lim_{n\to\infty} \sum_{k=1}^{K} (F, \psi_k)\, a_k^{(n)} = \sum_{k=1}^{K} (F, \psi_k) \lim_{n\to\infty} a_k^{(n)} = \sum_{k=1}^{K} (F, \psi_k)\, a_k \\ &= \left(F, \sum_{k=1}^{K} a_k \psi_k\right) = (F, f).\end{aligned}$$

This completes the proof of Theorem 1.

Note that each series $G = \sum_{k=1}^{\infty} b_k \psi_k$ (and hence each element of the space H) defines a linear functional on S by the following rule. Let

$$f = \sum_{k=1}^{K} f_k \psi_k \in S;$$

then, by definition, we set

$$G(f) = (G, f) = \sum_{k=1}^{K} b_k f_k. \tag{1.11}$$

The functional G is linear, since

$$G(\lambda f + \mu g) = \sum_{k=1}^{K} b_k \left(\lambda f_k + \mu g_k\right) = \lambda \sum_{k=1}^{K} b_k f_k + \mu \sum_{k=1}^{K} b_k g_k = \lambda G(f) + \mu G(g).$$

It follows from Theorem 1 that G is a continuous functional on S and hence a distribution on H. Now we note that the value of a distribution F on an element f in S is given by the relation

$$F(f) = (F, f) = \sum_{k=1}^{K} f_k \left(F, \psi_k\right). \tag{1.12}$$

Let us now justify the following assertion.

Theorem 2. *Let F be some distribution on H. Then the series*

$$G = \sum_{k=1}^{\infty} (F, \psi_k)\, \psi_k \tag{1.13}$$

is a distribution in S' and coincides with F.

Proof. Let f be an arbitrary element in S. Then, by definition, we have [see (1.11) and (1.12)]

$$G(f) = \sum_{k=1}^{K} (F, \psi_k)\, f_k = F(f).$$

Thus, we find that the set of arbitrary series of the form $\sum_{k=1}^{\infty} b_k \psi_k$ and the set of distributions on H coincide for this construction.

Now we define convergence in S' as the weak convergence of a sequence of functionals.

Definition 4. A sequence of distributions $F_1, F_2, \ldots, F_n, \ldots$ in S' converges to a distribution F in S' if $(F_n, f) \to (F, f)$ as $n \to \infty$ for each vector f in S. In this case, we write $F_n \to F$, $n \to \infty$, in S'.

The linear set S' with the introduced convergence is referred to as the *space S' of distributions*. Now one can prove the following assertion.

Theorem 3. *Let $F_1, F_2, \ldots, F_n, \ldots$ be a sequence in S' such that, for each vector f in S, the numerical sequence (F_n, f) is convergent as $n \to \infty$. Then the functional F defined on S by the relation*

$$(F, f) = \lim_{n\to\infty} (F_n, f)\,, \qquad f \in S, \tag{1.14}$$

is also linear and continuous on S; i.e., $F \in S'$.

Proof. By virtue of Theorem 1, it suffices to show that F is a linear functional. Indeed, we have

$$\begin{aligned}(F, \alpha f + \beta g) &= \lim_{n\to\infty} (F_n, \alpha f + \beta g) = \lim_{n\to\infty} (\alpha\,(F_n, f) + \beta\,(F_n, g)) \\ &= \alpha \lim_{n\to\infty} (F_n, f) + \beta \lim_{n\to\infty} (F_n, g) = \alpha(F, f) + \beta(F, g),\end{aligned}$$

where α and β are arbitrary numbers and f and g are elements of the space S.

It follows from Theorem 3 that the space S' of distributions is complete with respect to the introduced convergence.

2. SINGULAR INTEGRAL EQUATIONS ON THE CLASS OF PERIODIC FUNCTIONS

In applications [1, 9–11], problems are often reduced to the solution of linear singular integral equations of the first kind in the class of 2π-periodic functions, for which the corresponding characteristic equations have the form

$$L\,(g(\varphi), \varphi_0) = \frac{1}{\pi} \int_0^{2\pi} \ln \left| \sin \frac{\varphi_0 - \varphi}{2} \right| g(\varphi) d\varphi = f\,(\varphi_0)\,, \qquad \varphi_0 \in [0, 2\pi], \tag{2.1}$$

$$S\,(g(\varphi), \varphi_0) = \frac{1}{2\pi} \int_0^{2\pi} \cot \frac{\varphi_0 - \varphi}{2} g(\varphi) d\varphi = f\,(\varphi_0)\,, \qquad \varphi_0 \in [0, 2\pi], \tag{2.2}$$

$$H\,(g(\varphi), \varphi_0) = \frac{1}{4\pi} \int_0^{2\pi} \frac{g(\varphi) d\varphi}{\sin^2\left((\varphi_0 - \varphi)/2\right)} = f\,(\varphi_0)\,, \qquad \varphi_0 \in [0, 2\pi], \tag{2.3}$$

where $f(\varphi)$ is a 2π-periodic function.

By investigating the spectral properties of the operators $L\left(g(\varphi),\varphi_0\right)$, $S\left(g(\varphi),\varphi_0\right)$, and $H\left(g(\varphi),\varphi_0\right)$ with respect to the functions $\cos n\varphi$ and $\sin n\varphi$, $n=0,1,\dots$ [12], we obtain the relations

$$\frac{1}{\pi}\int\limits_0^{2\pi}\ln\left|\sin\frac{\varphi_0-\varphi}{2}\right|e^{in\varphi}d\varphi=\begin{cases}-(1/n)\operatorname{sgn}(n)e^{in\varphi_0} & \text{for}\quad n=\pm1,\pm2,\dots\\ -2\ln 2 & \text{for}\quad n=0,\end{cases}\tag{2.4}$$

$$\frac{1}{2\pi}\int\limits_0^{2\pi}\cot\frac{\varphi_0-\varphi}{2}e^{in\varphi}\,d\varphi=-i\operatorname{sgn}(n)e^{in\varphi_0},\qquad n=0,\pm1,\pm2,\dots,\tag{2.5}$$

$$\frac{1}{4\pi}\int\limits_0^{2\pi}\frac{e^{in\varphi}d\varphi}{\sin^2\left(\left(\varphi_0-\varphi\right)/2\right)}=-n\operatorname{sgn}(n)e^{in\varphi_0},\qquad n=0,\pm1,\pm2,\dots\tag{2.6}$$

From relations (2.4)–(2.6), we obtain the following. If we choose an arbitrary series of exponentials and formally apply one of the operators L, S, and H to it, then we obtain a similar series whose coefficients differ from the coefficients of the original series by a factor of the order of $\underline{n}^\lambda$, $\underline{n}^\lambda=\max\{1,|n|\}$. Thus, it is a natural idea to use the spaces H^λ similar to Sobolev spaces of distributions [4, 5]. To this end, we consider the set of complex-valued 2π-periodic functions on $[0,2\pi]$ with square-integrable absolute value. In this set of functions, one can introduce the inner product $(f,g)=\int_0^{2\pi}f(\varphi)\overline{g(\varphi)}\,d\varphi$ and the norm $\|f\|_2=\sqrt{(f,f)}$. We obtain the Hilbert space L_2 of complex-valued 2π-periodic functions on $[0,2\pi]$ with square integrable absolute value. In this space, an orthonormal basis is given by the function system $\psi_n=(2\pi)^{-1/2}e^{in\varphi}$, $n=0,\pm1,\pm2,\dots$ Now let $f(\varphi)$ be a 2π-periodic distribution on L_2; then

$$f(\varphi)=\sum_{n\in Z}\tilde f(n)\psi_n,\qquad \tilde f(n)=\left(f(\varphi),\bar\psi_n\right)=\frac{1}{\sqrt{2\pi}}\int\limits_0^{2\pi}f(\varphi)e^{-in\varphi}d\varphi,\tag{2.7}$$

where $\left(f(\varphi),\bar\psi_n\right)$ is treated as the value of a distribution (a linear functional) $f(\varphi)$ on the function $\bar\psi_n=(2\pi)^{-1/2}e^{-in\varphi}$. By H^λ we denote the set of distributions $f(\varphi)$ such that

$$\|f\|_\lambda=\left(\sum_{n\in Z}\underline{n}^{2\lambda}\left|\tilde f(n)\right|^2\right)^{1/2}<\infty;\tag{2.8}$$

i.e., the function $f^*(\varphi)=\sum_{n\in Z}\underline{n}^\lambda\tilde f(n)\psi_n(\varphi)$ belongs to the space L_2. Now, for functions in H^λ, we introduce the inner product by the relation

$$(f,g)=\sum_{n\in Z}\underline{n}^{2\lambda}\tilde f(n)\tilde g(n).\tag{2.9}$$

This inner product makes H^λ a Hilbert space; moreover, $H^0=L_2$ is the space of functions with square-integrable absolute value on $[0,2\pi]$.

Now for each distribution $g(\varphi)=\sum_{n\in Z}\tilde g(n)\psi_n$, we assume that

$$\begin{aligned}L\left(g(\varphi),\varphi_0\right)&=\frac{1}{\pi}\int\limits_0^{2\varphi}\ln\left|\sin\frac{\varphi_0-\varphi}{2}\right|g(\varphi)d\varphi\\&=-\sum_{n\in Z_0}\frac{1}{n}\operatorname{sgn}(n)\tilde g(n)\psi_n\left(\varphi_0\right)-2\ln 2\tilde g(0)\psi_0\left(\varphi_0\right),\qquad Z_0=\pm1,\pm2,\dots,\end{aligned}\tag{2.10}$$

$$S\left(g(\varphi),\varphi_0\right)=\frac{1}{2\pi}\int\limits_0^{2\varphi}\cot\frac{\varphi_0-\varphi}{2}g(\varphi)d\varphi=-\sum_{n\in Z}i\operatorname{sgn}(n)\tilde g(n)\psi_n\left(\varphi_0\right),\tag{2.11}$$

$$H\left(g(\varphi),\varphi_0\right)=\frac{1}{4\pi}\int\limits_0^{2\varphi}\frac{1}{\sin^2\left((\varphi_0-\varphi)/2\right)}g(\varphi)d\varphi=-\sum_{n\in Z}n\,\mathrm{sgn}(n)\tilde{g}(n)\psi_n\left(\varphi_0\right). \tag{2.12}$$

From (2.10)–(2.12), one can make the following conclusions. Equation (2.1) has a unique solution for any distribution $f(\varphi)$, Eqs. (2.2) and (2.3) have solutions neglecting a constant, and their solvability condition is given by the relation

$$\int\limits_0^{2\pi} f(\varphi)d\varphi=0, \tag{2.13}$$

where the value of the integral of $f(\varphi)$ over $[0,2\pi]$ is understood as the quantity $\sqrt{2\pi}\,\tilde{f}(0)$.

For these equations, we obtain the inversion formulas

$$g(\varphi)=\frac{1}{4\pi}\int\limits_0^{2\pi}\frac{f\left(\varphi_0\right)}{\sin^2\left((\varphi-\varphi_0)/2\right)}d\varphi_0-\frac{1}{2\ln 2}\int\limits_0^{2\pi}f(\varphi)d\varphi \tag{2.14}$$

for Eq. (2.1),

$$g(\varphi)=-\frac{1}{2\pi}\int\limits_0^{2\pi}\cot\frac{\varphi-\varphi_0}{2}f\left(\varphi_0\right)d\varphi_0+\frac{C}{2\pi} \tag{2.15}$$

for Eq. (2.2), and

$$g(\varphi)=\frac{1}{\pi}\int\limits_0^{2\pi}\ln\left|\sin\frac{\varphi-\varphi_0}{2}\right|f\left(\varphi_0\right)d\varphi_0+\frac{C}{2\pi} \tag{2.16}$$

for Eq. (2.3); in addition, C in (2.15) and (2.16) satisfies the relation

$$\int\limits_0^{2\pi} g(\varphi)d\varphi=C, \tag{2.17}$$

since condition (2.13) should be satisfied for the solvability of these equations. However, to construct numerical methods for Eqs. (2.1)–(2.3), it is necessary to estimate the closeness of solutions of these equations via the closeness of their right-hand sides. To this end, it is useful to note that the operator L is a mapping of the space H^λ into the space $H^{\lambda+1}$, the operator S is a mapping of the space H^λ into H^λ, and the operator H is a mapping of the space H^λ into $H^{\lambda-1}$; moreover, the operator L is a one-to-one mapping, and the kernels of the operators S and H consist of a set of constants. The considerations carried out in this section permit one to solve the following problem in a simple way. The problem on a circulation-free ideal incompressible flow past a circular cylinder with an external flow suction device can be reduced to the solution of the equation [4]

$$\frac{1}{2\pi}\int\limits_0^{2\pi}\cot\frac{\varphi_0-\varphi}{2}g(\varphi)d\varphi=\delta\left(\varphi_0-q\right)-\frac{1}{2\pi},\qquad \varphi_0,q\in[0,2\pi], \tag{2.18}$$

where the function $\delta(\varphi-q)$ is defined as the 2π-periodic delta function given by the relations $\delta(\varphi-q)=0$ for $\varphi\neq q$ and $\delta(\varphi-q)=+\infty$ for $\varphi=q$, $\varphi,q\in[0,2\pi]$, and $\int_0^{2\pi}\delta(\varphi-q)d\varphi=1$; i.e., $\int_0^{2\pi}\delta(\varphi-q)\psi_n(\varphi)d\varphi=\psi_n(q)$ for any function $\psi_n(\varphi)$, $n=0,\pm1,\pm2,\ldots$ Hence it follows that if a distribution $w(\varphi)$ is continuous in some neighborhood of the point q, then

$$\int\limits_0^{2\pi}\delta(\varphi-q)w(\varphi)d\varphi=w(q).$$

Now from (2.15), we find that the solution of Eq. (2.18) is given by the function

$$g(\varphi) = -\frac{1}{2\pi}\cot\frac{\varphi - q}{2} + \frac{C}{2\pi}. \tag{2.19}$$

Therefore, we obtain [4]

$$-\frac{1}{2\pi}\int_0^{2\pi}\cot\frac{\varphi_0-\varphi}{2}\frac{1}{2\pi}\cot\frac{\varphi-q}{2}d\varphi = \delta\left(\varphi_0 - q\right) - \frac{1}{2\pi}, \qquad \varphi_0, q \in [0, 2\pi]. \tag{2.20}$$

The same result can be obtained [4] if the distributions $\delta(\varphi - q)$ and $(2\pi)^{-1}\cot((\varphi - q)/2)$ are represented by Fourier series in the functions $\psi_n = (2\pi)^{-1/2}e^{in\varphi}$ and are substituted into Eq. (2.18). It follows from the representations by Fourier series that both above-mentioned functions belong to any space H^λ for $\lambda < -1/2$. Note also the following. Relations (2.4) and (2.5) imply that Eq. (2.2) can be integrated term by term; therefore,

$$-\frac{1}{\pi}\int_0^{2\pi}\ln\left|\sin\frac{\varphi_0-\varphi}{2}\right|\frac{1}{2\pi}\cot\frac{\varphi-q}{2}d\varphi = -\frac{\varphi_0}{2\pi} + F\left(\delta\left(\varphi_0 - q\right)\right) + C_q, \qquad \varphi_0, q \in [0, 2\pi], \tag{2.21}$$

where $F(\delta(\varphi - q)) = 0$, $0 \le \varphi < q$; $F(\delta(\varphi - q)) = 1/2$, $\varphi = q$; $F(\delta(\varphi - q)) = 1$, $q < \varphi \le 2\pi$ and $0 < q < 2\pi$; $F(\delta(\varphi)) = 0$ and $\varphi = 0$ for $q = 0$; $F(\delta(\varphi)) = 1/2$, $0 < \varphi < 2\pi$; and $F(\delta(\varphi)) = 1$, $\varphi = 2\pi$. Since the constant term in the Fourier series of the function occurring on the right-hand side in (2.21) should vanish, we have $C_q = (2\pi)^{-1}(q - \pi)$. Note that the function $F(\delta(\varphi - q))$ is an antiderivative of the function $\delta(\varphi - q)$ and vanishes for $\varphi = 0$.

The solution of the problem on the computation of the input resistance of a thin wire antenna powered by a power source leads [7] to the investigation of a hypersingular integral equation on an interval, whose right-hand side is a function with a singularity of the type of $1/x$ inside the solution domain. An analog of this equation in the periodic case is given by the equation

$$\frac{1}{4\pi}\int_0^{2\varphi}\frac{1}{\sin^2\left(\left(\varphi_0-\varphi\right)/2\right)}g(\varphi)d\varphi = \cot\frac{\varphi_0 - q}{2}, \qquad \varphi_0, q \in [0, 2\pi]. \tag{2.22}$$

By virtue of the inversion formula (2.16) for the considered equation and formula (2.21), the solution of Eq. (2.22) is given by the function

$$g(\varphi) = \frac{\varphi}{2\pi} - F(\delta(\varphi - q)) + C_q + \frac{C}{2\pi}, \qquad \varphi \in [0, 2\pi], \tag{2.23}$$

which satisfies relation (2.17).

For the approximate method of the solution of Eq. (2.20) by the discrete vortex method, it is important that the function $\delta(\varphi - q)$ is the limit of a sequence of 2π-periodic functions $\delta_h(\varphi - q) = 1/h$, $\varphi \in [q - h/2, q + h/2]$; $\delta_h(\varphi - q) = 0$, $\varphi \notin [q - h/2, q + h/2]$, $q \in [0, 2\pi]$, in the sense of Definition 4; i.e., $\lim_{h\to 0}\delta_h(\varphi - q) = \delta(\varphi - q)$.

3. SINGULAR INTEGRAL EQUATIONS ON THE INTERVAL $[-1, 1]$

Now we consider singular integral equations of the first kind on the interval $[-1, 1]$, to which numerous applied problems can be reduced [1, 9–11]. In this case, the characteristic equations can be represented in the form

$$L_{\varrho_1}\left(g(x), x_0\right) = \frac{1}{\pi}\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}\ln\left|x_0 - x\right|g(x)\,dx = f\left(x_0\right), \qquad x_0 \in (-1, 1), \tag{3.1}$$

$$S_{\varrho_1}\left(g(x), x_0\right)=\frac{1}{\pi}\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}\frac{g(x)dx}{x_0-x}=f\left(x_0\right), \qquad x_0\in(-1,1), \tag{3.2}$$

$$S_{\varrho_2}\left(g(x), x_0\right)=\frac{1}{\pi}\int_{-1}^{1}\sqrt{1-x^2}\frac{g(x)dx}{x_0-x}=f\left(x_0\right), \qquad x_0\in(-1,1), \tag{3.3}$$

$$S_{\varrho_3}\left(g(x), x_0\right)=\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\frac{g(x)dx}{x_0-x}=f\left(x_0\right), \qquad x_0\in(-1,1), \tag{3.4}$$

$$S_{\varrho_4}\left(g(x), x_0\right)=\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\frac{g(x)dx}{x_0-x}=f\left(x_0\right), \qquad x_0\in(-1,1), \tag{3.5}$$

$$H_{\varrho_2}\left(g(x), x_0\right)=\frac{1}{\pi}\int_{-1}^{1}\sqrt{1-x^2}\frac{g(x)dx}{\left(x_0-x\right)^2}=f\left(x_0\right), \qquad x_0\in(-1,1), \tag{3.6}$$

where $\varrho_1(x)=1/\sqrt{1-x^2}$, $\varrho_2(x)=\varrho_1^{-1}(x)$, $\varrho_3(x)=\sqrt{(1-x)/(1+x)}$, and $\varrho_4(x)=\varrho_3^{-1}(x)$.

One can show that the operators $L_{\varrho_1}\left(g(x), x_0\right)$, $S_{\varrho_k}\left(g(x), x_0\right)$, $k=1,2,3,4$, and $H_{\varrho_2}\left(g(x), x_0\right)$ satisfy the spectral relations [12, 13]

$$\frac{1}{\pi}\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}\ln\left|x_0-x\right|T_n(x)dx=-\frac{1}{n}T_n\left(x_0\right), \qquad x_0\in(-1,1), \qquad n=1,2,\ldots, \tag{3.7}$$

$$\frac{1}{\pi}\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}\ln\left|x_0-x\right|dx=-\ln 2, \qquad x_0\in(-1,1), \qquad n=0, \tag{3.8}$$

$$\frac{1}{\pi}\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}\frac{T_n(x)dx}{x_0-x}=-U_{n-1}\left(x_0\right), \qquad x_0\in(-1,1), \qquad n=0,1,2,\ldots, \tag{3.9}$$

$$\frac{1}{\pi}\int_{-1}^{1}\sqrt{1-x^2}\frac{U_{n-1}(x)dx}{x_0-x}=T_n\left(x_0\right), \qquad x_0\in(-1,1), \qquad n=1,2,\ldots, \tag{3.10}$$

$$\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\frac{P_n(x)dx}{x_0-x}=-Q_n\left(x_0\right), \qquad x_0\in(-1,1), \qquad n=0,1,2,\ldots, \tag{3.11}$$

$$\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\frac{Q_n(x)dx}{x_0-x}=P_n\left(x_0\right), \qquad x_0\in(-1,1), \qquad n=0,1,2,\ldots, \tag{3.12}$$

$$\frac{1}{\pi}\int_{-1}^{1}\sqrt{1-x^2}\frac{U_n(x)dx}{\left(x_0-x\right)^2}=-(n+1)U_n\left(x_0\right), \qquad x_0\in(-1,1), \qquad n=0,1,2,\ldots, \tag{3.13}$$

where

$$T_n(x)=\cos(n\arccos x), \qquad U_n(x)=\sin((n+1)\arccos x)/\sin(\arccos x),$$
$$P_n(x)=\left[T_{n+1}(x)-T_n(x)\right]/(1-x), \qquad Q_n(x)=U_n(x)-U_{n-1}(x).$$

For forthcoming considerations, we first recall the notion of the space $L_{2,\varrho}$ of functions on the interval $[-1,1]$ of the axis OX whose absolute values are square integrable on this interval with weight $\varrho = \varrho(x)$, that is, functions $f(x)$, $x \in [-1,1]$, such that $\int_{-1}^{1} \varrho(x)|f(x)|^2 dx$, where $\varrho = \varrho(x) > 0$ almost everywhere on $[-1,1]$. In a natural way, we introduce the inner product of functions in this space as $(f(x), g(x)) = \int_{-1}^{1} \varrho(x) f(x) \overline{g(x)}\, dx$. This makes $L_{2,\varrho}$ a Hilbert space. Therefore, in it there exists an orthonormal system of functions $\psi_{n,\varrho}$, $n = 0,1,2,\ldots$, that is a basis in it; i.e., $\int_{-1}^{1} \varrho(x)\psi_{n,\varrho}(x)\overline{\psi_{m,\varrho}(x)}\, dx = \delta_{n,m}$, where $\delta_{n,m}$ is the Kronecker delta. Any function $f(x) \in L_{2,\varrho}$ can be represented by a Fourier series in the system of these functions, which is convergent in the norm of $L_{2,\varrho}$:

$$f(x) = \sum_{n=0}^{\infty} \tilde{f}(n)\psi_{n,\varrho}(x), \qquad \tilde{f}(n) = \int_{-1}^{1} \varrho(x) f(x)\overline{\psi_{n,\varrho}(x)}\, dx, \tag{3.14}$$

$$\|f\|_{L_{2,\varrho}}^2 = \int_{-1}^{1} \varrho(x)|f(x)|^2 dx = \sum_{n=0}^{\infty} \left|\tilde{f}(n)\right|^2 < +\infty.$$

In the space $L_{2,\varrho}$, a complete orthonormal system is formed by the functions $T_n^{\bullet}(x) = \sqrt{2/\pi}\, T_n(x)$, $n = 1,2,\ldots$, and $T_0^{\bullet}(x) = \sqrt{1/\pi}\, T_0(x)$, for $\varrho = \varrho_1(x)$; the functions $U_n^{\bullet}(x) = \sqrt{2/\pi}\, U_n(x)$, $n = 0,1,2,\ldots$, for $\varrho = \varrho_2(x)$; the functions $P_n^{\bullet}(x) = \sqrt{1/\pi}\, P_n(x)$ for $\varrho = \varrho_3(x)$; and the functions $Q_n^{\bullet}(x) = \sqrt{1/\pi}\, Q_n(x)$ for $\varrho = \varrho_3(x)$.

Now let $f(x)$ be a distribution on $L_{2,\varrho}$ in the sense of Section 1; i.e., $f(x) \in S'_{\varrho} = S'(L_{2,\varrho})$; then

$$f(x) = \sum_{n=0}^{\infty} \bar{f}(n)\psi_{n,\varrho}(x), \qquad \bar{f}(n) = \left(f(x), \overline{\psi_{n,\varrho}(x)}\right) = \int_{-1}^{1} \varrho(x) f(x)\overline{\psi_{n,\varrho}(x)}\, dx, \tag{3.15}$$

where $\left(f(x), \overline{\psi_{n,\varrho}(x)}\right)$ is treated as the value of the distribution (linear functional) $f(x)$ on the function $\overline{\psi_{n,\varrho}(x)}$. By H_{ϱ}^{λ} we denote the set of distributions $f(x)$ on $L_{2,\varrho}$ such that

$$\|f\|_{\lambda,\varrho} = \left(\sum_{n\in Z} \underline{n}^{2\lambda} \left|\tilde{f}(n)\right|^2\right)^{1/2} < \infty; \tag{3.16}$$

i.e., the function $f^*(x) = \sum_{n\in Z} \underline{n}^{\lambda} \tilde{f}(n)\psi_{n,\varrho}(x)$ belongs to the set $L_{2,\varrho}$.

Now for functions in H_{ϱ}^{λ}, we introduce the inner product by the relation

$$(f,g) = \sum_{n\in Z} \underline{n}^{2\lambda} \tilde{f}(n)\overline{\tilde{g}(n)}. \tag{3.17}$$

The inner product thus defined makes H_{ϱ}^{λ} a Hilbert space; moreover, $H_{\varrho}^{0} = L_{2,\varrho}$ is the space of function with square integrable absolute value on $[-1,1]$. Now for each distribution

$$g(x) = \sum_{n=0}^{\infty} \tilde{g}(n)\psi_{2,\varrho}(x), \qquad \varrho(x) = \varrho_k(x), \qquad k = 1,2,3,4,$$

we assume that the values $L_{\varrho_1}(g(x), x_0)$, $S_{\varrho_k}(g(x), x_0)$, $k = 1,2,3,4$, and $H_{\varrho_2}(g(x), x_0)$ are obtained by interchanging the integral and sum in these operators. Now relations (3.7)–(3.13) imply the following. The operator $L_{\varrho_1}(g(x), x_0)$ is a one-to-one mapping of the space of distributions $S'(L_{2,\varrho_1})$ into itself and the space $H_{\varrho_1}^{\lambda}$ into the space $H_{\varrho_1}^{\lambda+1}$; the operator $S_{\varrho_1}(g(x), x_0)$ is a mapping of $S'(L_{2,\varrho_1})$ onto $S'(L_{2,\varrho_2})$ (the kernel of the mapping is a set of constants) and is a mapping of $H_{\varrho_1}^{\lambda}$

onto $H^{\lambda}_{\varrho_2}$ with the same property; the operator $S_{\varrho_2}(g(x), x_0)$ is a mapping of $S'(L_{2,\varrho_2})$ into $S'(L_{2,\varrho_1})$, the functions $f(x)$ in its range satisfy the relation

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx = 0, \tag{3.18}$$

and it is a mapping of $H^{\lambda}_{\varrho_2}$ into $H^{\lambda}_{\varrho_1}$ with the same property; the operator $S_{\varrho_3}(g(x), x_0)$ is a one-to-one mapping of $S'(L_{2,\varrho_3})$ into $S'(L_{2,\varrho_4})$ and of $H^{\lambda}_{\varrho_3}$ into $H^{\lambda}_{\varrho_4}$; the operator $S_{\varrho_4}(g(x), x_0)$ is a one-to-one mapping of $S'(L_{2,\varrho_4})$ into $S'(L_{2,\varrho_3})$ and of $H^{\lambda}_{\varrho_4}$ into $H^{\lambda}_{\varrho_3}$; and finally, the operator $H_{\varrho_2}(g(x), x_0)$ is a one-to-one mapping of $S'(L_{2,\varrho_2})$ into $S'(L_{2,\varrho_2})$ and of $H^{\lambda}_{\varrho_2}$ into $H^{\lambda-1}_{\varrho_2}$. Now we make the following important remark. The well-known inversion formulas [7, 12, 14] are valid for Eqs. (3.1)–(3.6) on basis elements of the corresponding spaces. Therefore, by virtue of the definition of values of the operators $L_{\varrho_1}(g(x), x_0)$, $S_{\varrho_k}(g(x), x_0)$, $k = 1, 2, 3, 4$, and $H_{\varrho_2}(g(x), x_0)$, the following inversion formulas are valid on distributions in the corresponding spaces for Eqs.(3.1)–(3.6):

$$g(x) = \frac{1}{\pi}\left(\int_{-1}^{1} \frac{1 - x x_0}{\sqrt{1-x_0^2}\,(x-x_0)^2} f(x_0)\, dx_0 - \frac{1}{\ln 2}\int_{-1}^{1} \frac{f(x)dx}{\sqrt{1-x^2}}\right), \qquad x \in (-1, 1),$$

for Eq. (3.1),

$$g(x) = -\frac{1}{\pi}\int_{-1}^{1} \frac{\sqrt{1-x_0^2} f(x_0)\, dx_0}{x - x_0} + \frac{C}{\pi}, \qquad x \in (-1, 1), \qquad \int_{-1}^{1} \frac{g(x)}{\sqrt{1-x^2}} dx = C \tag{3.19}$$

for Eq. (3.2),

$$g(x) = -\frac{1}{\pi}\int_{-1}^{1} \frac{f(x_0)\, dx_0}{\sqrt{1-x_0^2}\, x - x_0}, \qquad x \in (-1, 1),$$

for Eq. (3.3),

$$g(x) = -\frac{1}{\pi}\int_{-1}^{1} \sqrt{\frac{1+x_0}{1-x_0}} f(x_0)\, dx_0, \qquad x \in (-1, 1),$$

under condition (3.18) for Eq. (3.4),

$$g(x) = -\frac{1}{\pi}\int_{-1}^{1} \sqrt{\frac{1-x_0}{1+x_0}} f(x_0)\, dx_0, \qquad x \in (-1, 1),$$

for Eq. (3.5), and

$$g(x) = \frac{1}{\pi\sqrt{1-x^2}}\int_{-1}^{1} \ln\left|\frac{x - x_0}{1 - x x_0 + \sqrt{1-x^2}\,\sqrt{1-x_0^2}}\right| f(x_0)\, dx_0, \qquad x \in (-1, 1),$$

for Eq. (3.6).

As an example of the use of inversion formulas in the class of distributions, consider Eq. (3.2). On its right-hand side, we choose the function $f(x_0) = \delta(x_0 - q)$, $q \in (-1, 1)$; then, by virtue of (3.19), we find that the solution of this equation is given by the function ($C = 0$)

$$g(x) = -\frac{1}{\pi}\frac{\sqrt{1-q^2}}{x - q}.$$

Now, by substituting the last function into Eq. (3.2), we obtain

$$\frac{1}{\pi}\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}\,(x_0-x)}\left(-\frac{1}{\pi}\frac{\sqrt{1-q^2}}{x-q}\right)dx=\delta\left(x_0-q\right),\qquad x_0\in(-1,1).$$

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